All (!) Three-Part Variations on Three Different Kinds of Cubes

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Abstract

Sol LeWitt (1928-2007) made or designed several artworks whose titles contain the phrase "three-part variations on three different kinds of cubes." To a mathematician, these pieces suggest a combinatorics problem: How many variations are there? In this paper we present a new way of diagramming LeWitt's variations, we describe how to solve the combinatorics problem using Burnside's Counting Theorem, and we share some photos of our 3D-printed versions of the variations, including a new 57th variation.

Introduction

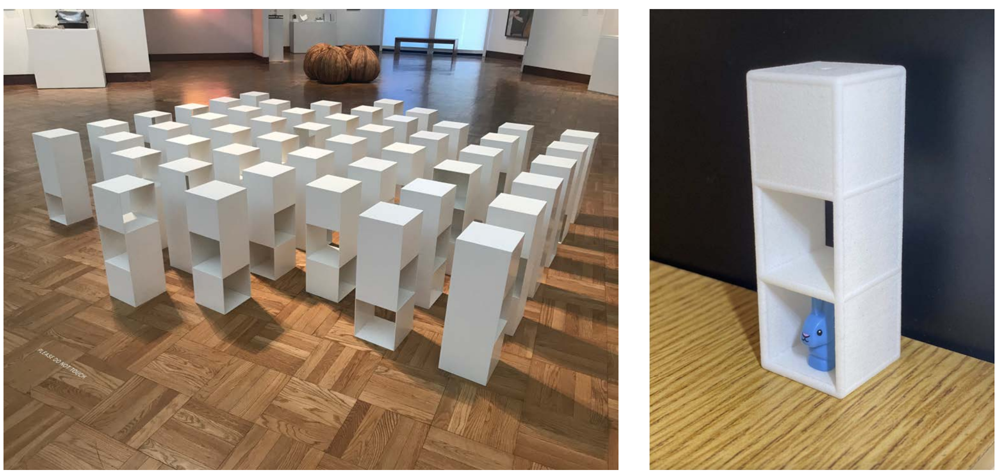

Figure 1: *(a) Sol LeWitt's "49 Three-Part Variations on Three Different Kinds of Cubes" at Oberlin's AMAM and (b) a 3D-printed version of LeWitt's variation* $(1, 2FB, 3F)$. *Photos by Robert Bosch.*

The Allen Memorial Art Museum (AMAM) in Oberlin, Ohio is proud that its collection includes an artwork by Sol LeWitt entitled *49 Three-Part Variations on Three Different Kinds of Cubes* [2]. LeWitt's piece is made of three different types of white enamel-coated steel cubes. A 1-CUBE is a solid cube; it has all six of its sides. A 2-CUBE has two opposite sides removed—either the front and back sides, or the right and left sides. A 3-CUBE has only one side removed—the front, back, left, or right. LeWitt placed copies of these cubes in towers of three. Figure 1a displays a photo of 49 variations arranged on the floor of the AMAM, while Figure 1b shows a 3D-printed version of the variation $(1, 2FB, 3F)$. The ordered triple notation uses three entries to specify the cube types, listing them in top-middle-bottom order. In the triple $(1, 2FB, 3F)$, the 1 tells us that the top cube is a 1-CUBE, the $2FB$ tells us that the middle cube is a 2-CUBE with its front and back sides removed, and the $3F$ tells us that the bottom cube is a 3-CUBE with its front side removed. In Figure 1b, the front is the side facing away from the black wall, and there's a plastic bunny in the 3-CUBE.

When we first encountered LeWitt's piece, we immediately contemplated the combinatorics problem it suggests: How many distinct variations are there? Most mathematicians would have the same reaction (and also wonder what it means for variations to be distinct). LeWitt seems to have been like-minded, as he made a hand-drawn diagram for a follow-up piece entitled *All Three-Part Variations on Three Different Kinds of Cubes* [3]. This follow-up piece has 56 variations. Figure 2 displays a diagram we created after viewing LeWitt's diagram. The top section presents the three different kinds of cubes. The bottom section shows all 56 of LeWitt's variations and how he wanted them to be arranged on the floor. In the bottom section, the top left subdiagram (highlighted in light gray) shows how we denote variation $(1, 2FB, 3F)$, the one shown in Figure 1b. When we read our subdiagrams, we start in the center and move outward. In the top left shaded subdiagram, the innermost portion is a square, which indicates that the variation's top cube has all four of its sides (and is therefore a 1-CUBE). The middle portion consists of two parallel vertical lines, and they indicate that the variation's middle cube has its front and back sides removed (and is therefore a 2-CUBE of type FB). The outermost portion is a square with its bottom side removed, which indicates that the variation's bottom cube has its front side removed (and is therefore a 3-CUBE of type F).

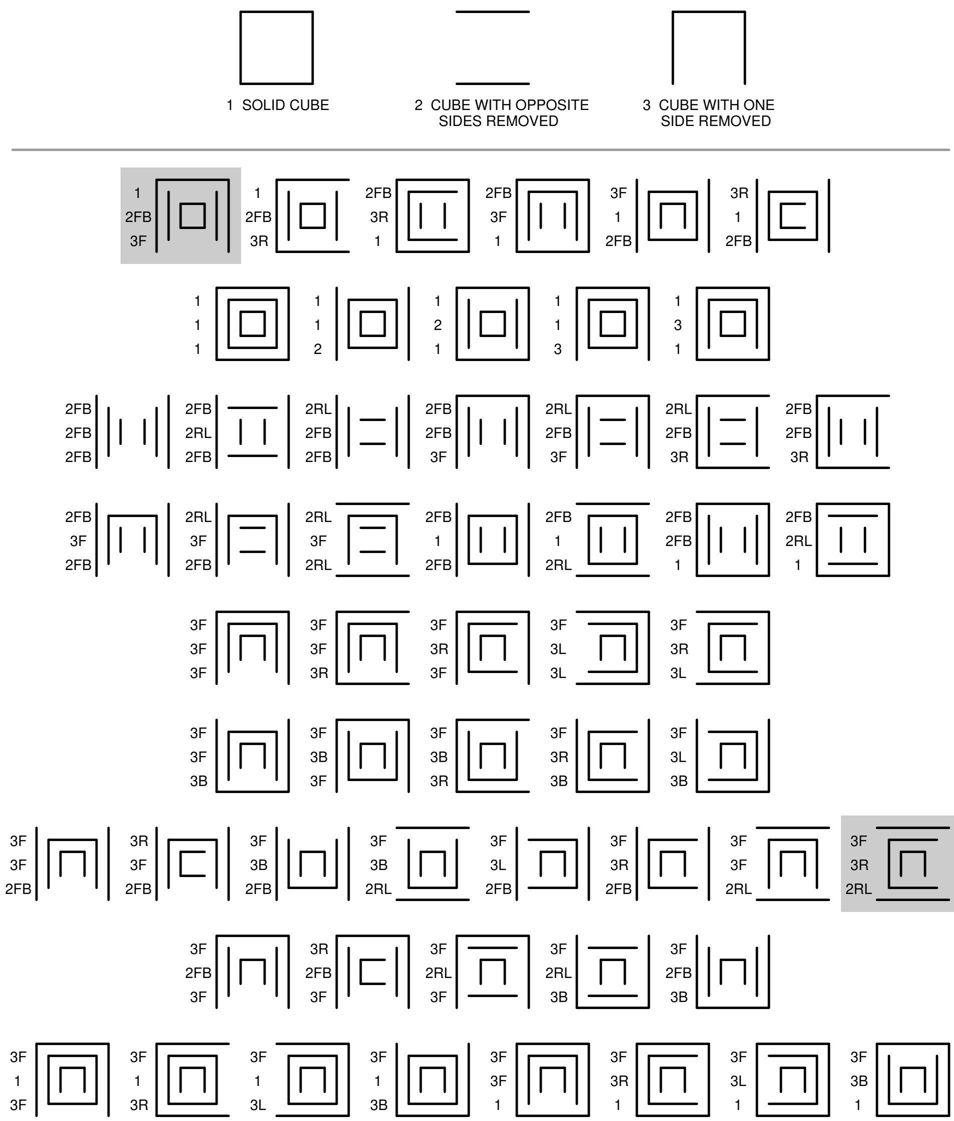

Figure 2: *Our diagram of LeWitt's "All Three-Part Variations on Three Different Kinds of Cubes."*

Counting Distinct Variations

A cursory examination of LeWitt's towers shows that LeWitt considered one variation to be distinct from another if neither can be obtained by rotating, or by rotating and then flipping over, the other. Consider variation $(3F, 3R, 2RL)$, which is highlighted in light gray in Figure 2. The first four subdiagrams in Figure 3 show what happens when we rotate $(3F, 3R, 2RL)$ by 0°, 90°, 180°, and 270° counterclockwise. The last four subdiagrams show what happens when we perform a "right-left flip" operation on the rotations. To execute a right-left flip, we face the tower from the front and grab the bottom cube with our right hand and the top cube with our left hand. We then pick up the entire tower, flip it upside down, and return it to the floor. The cube that was formerly the bottom cube is now the top cube, and the cube that was formerly the top cube is now the bottom cube. All eight of these subdiagrams correspond to the same object. The shaded one, variation $(3F, 3R, 2RL)$ is the one that LeWitt included in his diagram.

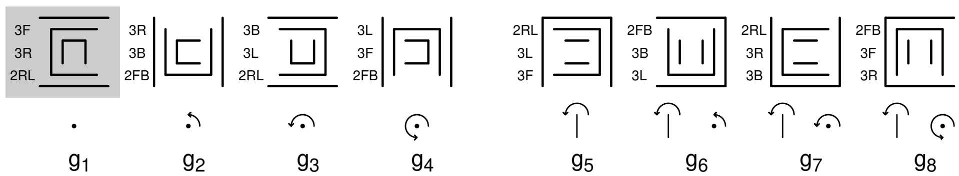

Figure 3: *The four rotations and the four right-left flips of the four rotations.*

The Burnside Counting Theorem is used to determine how many distinct objects exist in the presence of rotations and reflections [1]. Let S be any finite collection of objects, and let G be a finite group of symmetries for these objects. Suppose that G has n symmetry operators $g_1, g_2, g_3, \ldots, g_n$ and that g_1 is the identity operator. Let $C(g)$ denote the number of objects in the collection S that are fixed by the symmetry operator g of G. Then the number N of objects in S that are distinguishable relative to the symmetries of G is given by

$$N = \frac{1}{n}\left[C(g_1) + C(g_2) + C(g_3) + \ldots + C(g_n)\right].$$

For the Sol LeWitt towers of cubes, the symmetry operators are the four rotations and the four right-left-flipped rotations shown in Figure 3. So for the Sol LeWitt variations, $n = 8$.

We can calculate $C(g_1)$ by considering the three different kinds of cubes: The 1-cube has one form, the 2-cube has two possible forms for positioning purposes (FB and RL), and the 3-cube has four possible forms for positioning (F, B, R, and L), for a total of seven options. As we need to select an option for each layer (top, middle, and bottom), we end up with $C(g_1) = 7^3 = 343$.

To compute $C(g_2)$, as well as $C(g_4)$, we observe that neither 2-cubes nor 3-cubes are left fixed by a 90° rotation or a 270° rotation. Accordingly, the only object left fixed by either g_2 or g_4 is variation $(1, 1, 1)$, the variation made entirely of 1-cubes. Hence $C(g_2) = C(g_4) = 1$.

To compute $C(g_3)$, we note that 1-cubes and 2-cubes are fixed by a 180° rotation, but 3-cubes are not, so our only options are the variations made of 1-cubes and 2-cubes, of which there are $(1 + 2)^3 = 27$. As a result, we end up with $C(g_3) = 27$.

To compute $C(g_5)$, we observe that we have five potential choices for the middle layer: the 1-cube, either form of the 2-cube (FB or RL), and two of the four forms of the 3-cube (F or B, but not R or L). All seven options are available for the top layer, but once the top cube has been selected, there will be only one choice that works for the bottom layer. (The seven possible top/bottom pairs are $1/1$, $2FB/2FB$, $2RL/2RL$, $3F/3F$, $3B/3B$, $3R/3L$, and $3L/3R$.) As a consequence, we have $C(g_5) = 5 \cdot 7 = 35$.

To compute $C(g_6)$, we note that the middle layer must be a 1-cube. All seven options are available for the top layer, but as with g_5, once the top cube has been selected, there will be only one choice that works for the bottom layer. (The seven possible top/bottom pairs are $1/1$, $2FB/2RL$, $2RL/2FB$, $3F/3L$, $3B/3R$, $3R/3B$, and $3L/3F$.) Consequently, we end up with $C(g_6) = 1 \cdot 7 = 7$.

The analysis for $C(g_7)$ is similar to that of $C(g_5)$, and the analysis for $C(g_8)$ is similar to that of $C(g_6)$. We have $C(g_7) = C(g_5) = 35$ and $C(g_8) = C(g_6) = 7$. And now that we have the values of $C(g_1)$ through $C(g_8)$, we can substitute them into the Burnside Counting Theorem equation:

$$\begin{aligned} N &= \frac{1}{n}\left[C(g_1) + C(g_2) + C(g_3) + \ldots + C(g_n)\right] \\ &= \frac{1}{8}\left[343 + 1 + 27 + 1 + 35 + 7 + 35 + 7\right] = 57, \end{aligned}$$

which means that there are 57 distinct variations, not 56 as LeWitt proposed!

By going through all of the variations systematically, we were able to determine that the variation that LeWitt missed can be denoted $(3F, 2RL, 3R)$, but there are three other ways to describe it. (Each right-left flip of one of the four rotations produces another one of the four non-right-left-flipped rotations.) This variation is the 57th variation, completing ALL three-part variations on three different kinds of cubes. Figure 4 displays a photo of a 3D-printed version, with a plastic bunny inside each of the 3-CUBES.

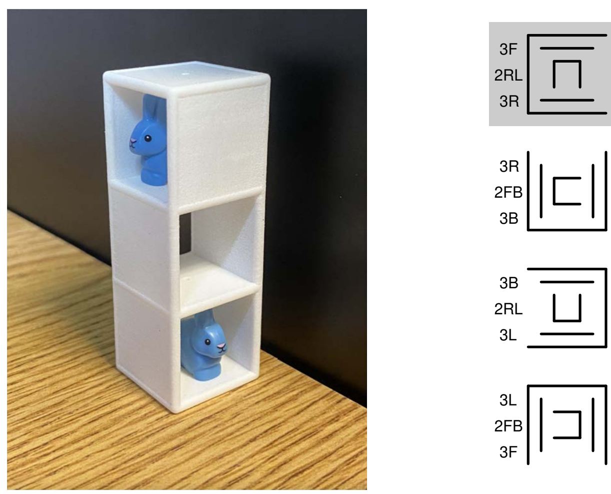

Figure 4: *Variation* $(3F, 2RL, 3R)$*, the* 57*th and final variation. Photo by Robert Bosch.*

Acknowledgements

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References

[1] Solomon W. Golomb. *Polyominoes: Puzzles, Problems, Patterns, and Packings*. Princeton University Press, 1996.

[2] Sol LeWitt. *49 Three-Part Variations on Three Different Kinds of Cubes*, 1967-71. Enamel on steel. Allen Memorial Art Museum, Oberlin, Ohio.

[3] Sol LeWitt, Alecia Legg, Lucy R. Lippard, Bernice Rose, and Robert Rosenblum. *Sol LeWitt: the Museum of Modern Art, New York [Exhibition]*, Museum of Modern Art, 1978.