Pattern Continuity in Polygon Tessellations

Severi Virolainen

Aalto University, Helsinki, Finland; severi@algo.fi

Abstract

Polygonal tessellations create exciting and fascinating patterns. These tessellations (tilings) are used in architecture, textiles, board games, and various other fields for practical, structural, and decorative purposes. If the polygons (tiles) contain a pattern, and the patterns connect at the polygon's edges, the combined patterns can be even more fascinating. This study presents logic and constraints governing the appearance of polygons, with a primary focus on edges and a secondary focus on the patterns inside. Additionally, I question the practice of referring to patterned polygons as Truchet tiles.

Introduction

Numerous Bridges papers—some referred to in this paper—investigate polygon tessellations, where a group of polygons containing lines or patterns combine to form larger patterns—the pattern continues from polygon to polygon. These polygons are often called Truchet Tiles. Despite the many impressive creations and beautiful larger patterns discovered, the fundamental rules of these continuities have not been thoroughly explored.

Tiles of various colors can create stunning patterns. By using various tessellations, the possibilities are virtually limitless, even with single-color tiles and without considering the patterns inside. This paper concentrates on patterned tiles. Specifically, it examines how these patterns meet at the tile's edges and create larger patterns.

Numerous polygon tessellations exist [2]. However, due to space constraints, this paper is limited to regular polygons and tessellations, and flat surfaces. Therefore, it covers only triangles, squares, and hexagons, with a few exceptions.

Truchet Square

The illustrious Truchet square is a fascinating example of patterned polygons. Father Sébastien Truchet is credited with developing this two-colored square, divided diagonally into white and black sections.

Truchet squares can be used following pattern continuity rules, where matching edges align—meaning white edges meet white edges, and black edges meet black edges. Many tilings presented using Truchet squares do not adhere to this restriction. Figure 1 showcases Truchet tilings, both random and those following pattern continuity rules.

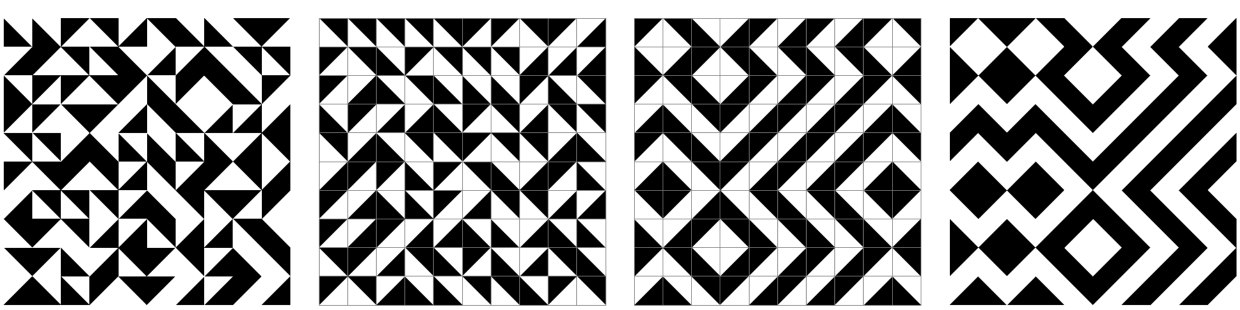

Figure 1: *Examples of random and pattern continuity tilings of Truchet squares.*

When adhering to the rules, Truchet squares are quite limiting. The first tile can have any of the four square rotations, but the second and third only have two, and the fourth has no options. In practical applications, the limitations would be even more pronounced. A logical tiling order would be to place a tile only when two adjacent edges are known. Otherwise, a careless tiler might encounter a situation where no more tiles can be added.

The greatest merit of the Truchet square lies in its simplicity: it comprises just one tile and one exceptionally simple pattern, yet it can be used to create various interesting patterns. Pelletier-Auger beautifully demonstrates this on his webpage [8].

Naming System

In this work I consider tilings that are similar to the Truchet pattern continuity tilings discussed above, except that I consider hexagonal and triangular tiles, along with squares, and edges can be multicolored. To ensure the polygonal tiles maintain pattern continuity, the multicolored edges of adjacent polygons must align, so I characterize polygons based on the color segmentation of their edges. The actual pattern within the polygon does not influence the continuity and is considered secondary.

An edge can be divided into segments. In this study, I utilize equidistant segments; however, other types of divisions also follow either asymmetric or symmetric logic, depending on their symmetry. I will first explain symmetric edges and then asymmetric edges.

Symmetrical Edges

Symmetrical edges consist of a symmetrical sequence of segments. Figure 2 provides examples of symmetrical edges.

Figure 2: *Different symmetrical edges. As the width of a segment does not affect symmetry, it is just an artistic choice, and a line is logically a narrow segment. Segments of different colors are marked with capital letters. The name of the edge is according to the letters. The names of the edges in the figure are A, ABA, ABA, and ABCBA, respectively.*

If only one segment exists, the edge has just one color (the first image in Figure 2), such as in the Truchet Tile (Figure 1). An edge with a single color is always symmetrical by nature. The second simplest symmetrical edge is divided into three segments, such as white-black-white. If it were red-green-blue, it would not be symmetrical. Any odd number of segments can be colored symmetrically, as demonstrated in Figure 2.

Additionally, an edge with a line starting from the middle should be classified as a symmetrical edge with three segments—the middle segment is as narrow as the line. The logic of continuity is the same for all polygons with symmetrical edges.

If the edge colorings of two tiles are symmetric and have the same segmentation structure, then they can connect to continue the pattern within the tiles. For example, an ABA square can seamlessly connect with an ABA hexagon or triangle. Figure 3 presents examples of tilings of polygons with symmetrical edges, both with and without polygon borders.

A polygon with symmetrical edges can have different symmetrical edges. Polygons whose number of edges is not a prime number may have alternating edges. For example, the first edge could be ABA and the next one BAB repeatedly. Such a polygon is named ABA-BAB, where the hyphen marks the corner. Another example is the Truchet square, which could be named A-A-B-B square, meaning two white edges followed by two black edges.

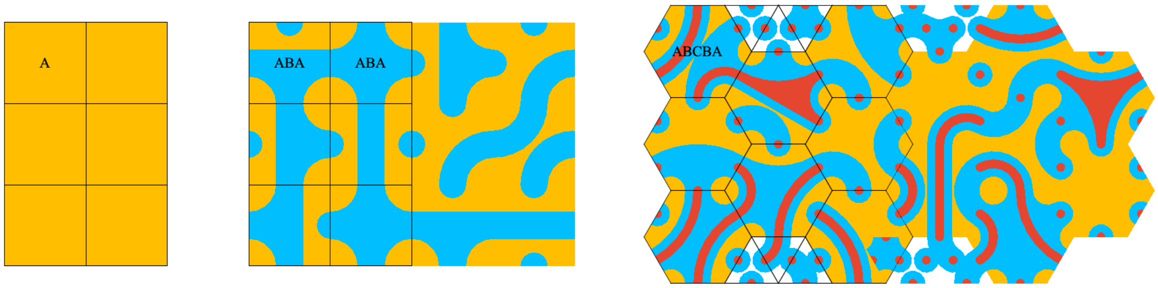

Figure 3: *These examples include a square, which is a simple monochromatic tile; an ABA square with random patterns inside; and an ABCBA hexagon, also with random patterns. Note the ABCBA triangles filling the gaps (with a white background to make them stand out).*

For efficiency, tiles with repeating edge sequences will be denoted by the fundamental string generating the repeating edge pattern. Thus, the ABA-ABA-ABA-ABA square is simply the "ABA square," but the ABA-ABA-BAB-BAB square must be named in a longer format. (Note that the ABA-ABA-BAB-BAB square is a "decorated" version of the Truchet square.)

ABA-BAB is impossible for triangles and other polygons with an odd number of edges since that segmentation cannot be repeated without a remainder. For squares, ABA-BAB works well, as visualized in Figure 4. However, ABA-BAB hexagons have a pattern continuity issue, as shown in Figure 4. This problem can be circumvented by using ABA hexagons so that every intersection has two ABA-BAB and one ABA hexagon.

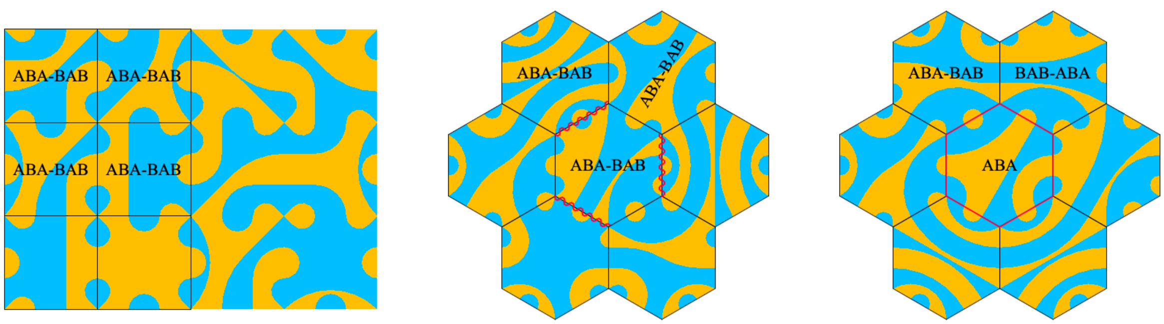

Figure 4: *ABA-BAB squares connect well, but it is important to note that all ABA edges are horizontal and all BAB edges are vertical (or vice versa). For hexagons, pure ABA-BAB tiling is not possible. Red wavy lines indicate problematic connections of edges. As demonstrated on the right, continuity can be resolved by incorporating ABA hexagons as part of the pattern.*

Asymmetrical Edges

When an edge has an even number of segments, it inevitably becomes asymmetrical. Such polygons could have AB or AB-BA edges, or even AC-BC, which needs to be combined with CB-AB and BA-CA in the case of hexagons. Perhaps the most popular asymmetrical example is the AB-BA square, which, for instance, Reimann has used for typing at 45 degrees [6] and Carlson has scaled in a highly creative way [1]. Both are exemplary examples of the usage of pattern continuity.

An asymmetrical edge requires a mirror edge to connect. AB connects to BA. More precisely, to create a continuous pattern, we need AB-BA and BA-AB squares in a checkerboard rhythm, as explained in more detail by Reimann [6]. BA-AB is actually AB-BA rotated 90 or 270 degrees, as well as AB-BA whose edges are shifted one step. Both AB-BA square patterns and all three different AB-BA hexagon patterns are shown in Figure 5.

Also, AB squares create a continuous pattern, but similar to AB-BA, every second square must be mirrored edge-wise. So every second square is BA, as seen in Figure 5.

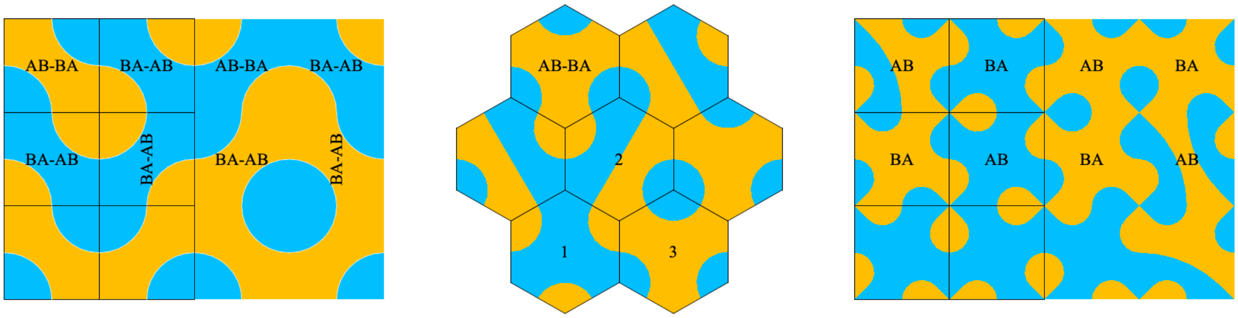

Figure 5: *AB-BA squares necessitate a checkerboard sequence through a 90-degree rotation or by shifting the pattern. Shifting AB-BA turns it to BA-AB. In contrast, AB-BA hexagons consistently maintain the same edge-wise rotation. AB squares also require a checkerboard sequence.*

AB hexagons have the same limitation as ABA-BAB hexagons in Figure 4. Similarly, AB hexagons could be used if every third hexagon is AB-BA. The rules for joining polygons with different edges are not discussed further in this work.

Mitchell visualizes similar polygons in his paper "Generalizations of Truchet Tiles" [4]. Presented polygons are mostly ABA—representing an edge with a line starting from the middle, and ABABA, with two lines starting from every edge. If the areas bounded by the line were colored and the line was forgotten, they would have different continuities. For example, ABA would turn into AB and ABABA into ABA. Later in the same paper, he colors ABAB squares and octagons. The pattern continuity rules of ABAB and AB are similar.

A pure AB-BA triangle is impossible, but AB has no problem. Triangles also follow a checkerboard pattern, where AB and BA triangles are placed alternately, as shown in Figure 6. Also, the AB-BA triangle can be used with a variable ending (AB or BA) in a checkerboard tiling with an additional limitation to the rotation.

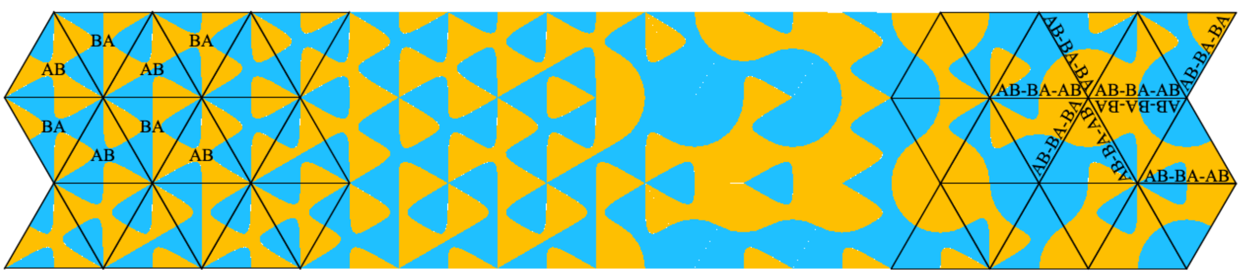

Figure 6: *Three different AB triangles on the left half create a minimalistic pattern. AB-BA-AB and AB-BA-BA triangles on the right half have more limited tiling.*

With a third color C, it is possible to create hexagonal tilings where every edge has two out of three colors in pairs. With AB-CB hexagons being used together with BC-AC and CA-BA hexagons, patterns continue seamlessly. These three differently colored hexagons are tiled in a hexagonal grid so that every corner has all three tiles in the correct rotation. Like AB-BA hexagons, there are three possible rotations (0, 120, and 240 degrees), as those three edges are identical. This logic is visualized in Figure 7.

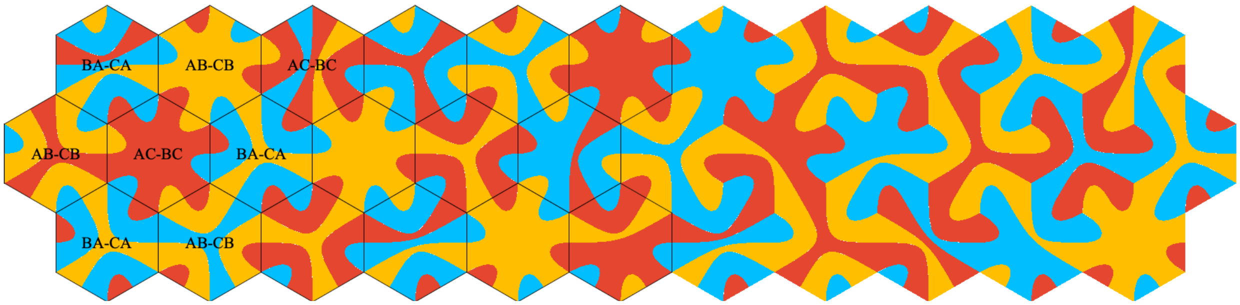

Figure 7: *AB-CB, BC-AC, and CA-BA hexagons tiled contain randomly chosen patterns inside.*

Asymmetric edges follow the same logic as a jigsaw puzzle. The edges and their opposites (mirrored) could be replaced with male and female-shaped edges. In the case of three-color hexagons, there would be three different pairs of male–female connections (AB to BA, BC to CB, and CA to AC).

Patterns and Larger Patterns

Edges define how polygons containing patterns (decorations) connect seamlessly, but the actual patterns determine how the larger pattern continues or discontinues inside the polygon. Generally, the pattern can have any appearance, which generates an infinite number of possible larger patterns. In this study, I use more or less minimalistic forms to create patterns inside polygons.

In the case of symmetric polygons, the simplest forms can be created with arcs and straight lines only, as seen in previous figures. In the case of asymmetric polygons, I use both combinations of arcs and straight lines, as well as Bézier curves. They have a slightly different appearance, and it is a matter of taste which one prefers. I prefer Bézier curves, but I haven't found mathematical reasoning for how sharp the curves should be. Some different styles are demonstrated in Figure 8. Mitchell goes even further into the freedom of patterns in his study [4].

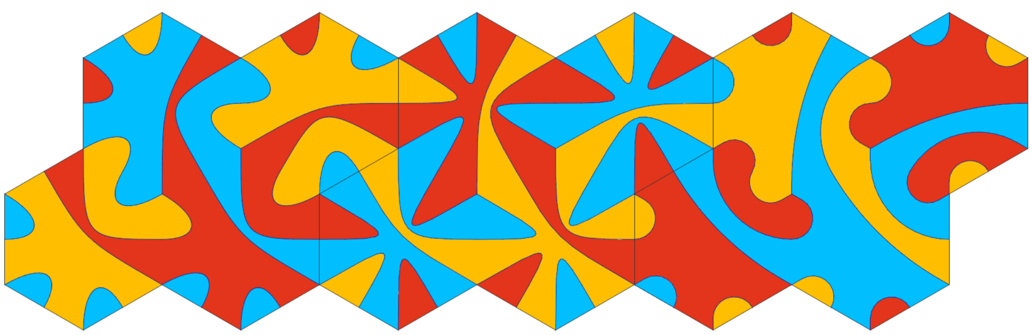

Figure 8: *Edge-wise identical hexagons in three styles tiled in three groups of four. The left one is the style that I have mostly used—mild Bézier curves. The middle one is more dynamic, and Bézier curves are sharper. On the right, patterns are formed by arcs and straight lines. All three groups have identical topologies of the patterns, with just the colors permutating.*

Patterns

The pattern inside the polygon determines which color continues to the next tiles and which branch of the larger pattern ends. Different polygons with different edge types have different numbers of possible pattern topologies. Asymmetrical patterns can be rotated, so they have more than one appearance. Exemplary quantities of appearances are presented in Table 1.

Table 1: *Numbers of different pattern topologies in polygons of different edge types. With rotations (w Rots) refers to the same pattern in different rotations in the grid. Some patterns have only one rotation, while others have as many as the polygon's edges. The label nc means that I have not calculated or counted them (yet). "—" stands for not possible. A pentagon could be used to form a dodecahedron with pattern continuity, and a square to form a cube. Interestingly, a square has six patterns to match the six sides of a cube.*

	Triangle		Square		Pentagon		Hexagon	
Edge type	Patterns	w Rots	Patterns	w Rots	Patterns	w Rots	Patterns	w Rots
Symmetrical								
A	1	1	1	1	1	1	1	1
ABA	3	5	6	14	10	42	28	138
ABCBA	6	12	19	55	nc	nc	255	nc
ABA-BAB	—	—	28	138	—	—	nc	nc
Asymmetrical								
AB	3	5	6	14	10	42	28	138
AB-BA	—	—	2	2	—	—	3	5
AB-CA-BC	1	1	—	—	—	—	nc	nc
AB-CB	—	—	nc	nc	—	—	4	12

The versatility of the ABA hexagon patterns and rotations, as demonstrated in Figure 9, allows for a wide range of applications. Limiting the alternatives to specific tiles and rotations can generate more controlled or regular patterns. This approach is presented in more detail [8], showcasing the potential of such patterns in various contexts.

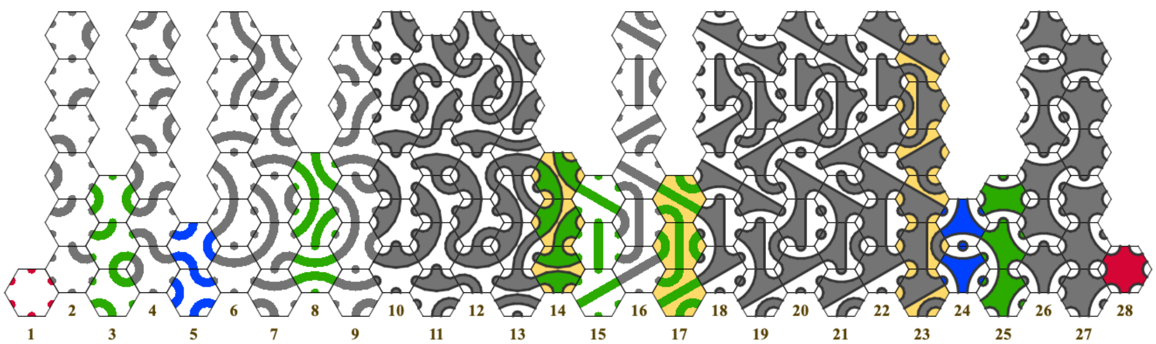

Figure 9: *The ABA hexagons feature 28 topologies with 138 appearances due to rotations for color B patterns (gray, green, blue, or red). Red patterns have one rotation, green patterns have three, and blue patterns have two. A yellow background indicates no dead-ends, and thick outlines signify forking. This variety allows for intricate patterns and designs. To be complete: "the backgrounds" (white and yellow) have the same number of topologies.*

One such application is the creation of letters and written artwork [7]. By designing specific patterns and rotations within the ABA hexagon system, one can form recognizable characters or symbols, such as text within the pattern. I have incorporated ABA hexagons into various artworks, including a mural in the Väre building at Aalto University [7]. You can find several applications of this concept in my earlier paper [8] and one example in Figure 12.

Three-dimensional effects are also common, and they often resemble ribbons that overlap and intertwine. One such example is the hexagon with three ribbons, two of which cross each other, presented by Katz [3]. Ribbon patterns typically exhibit symmetrical edge logic, which allows them to be connected in various ways and even mirrored. In highly regular patterns, the ribbons could have different colors; however, pattern continuity becomes impossible when rotation is random. Of course, ribbons can still be hand-colored afterward, as demonstrated in Figure 10.

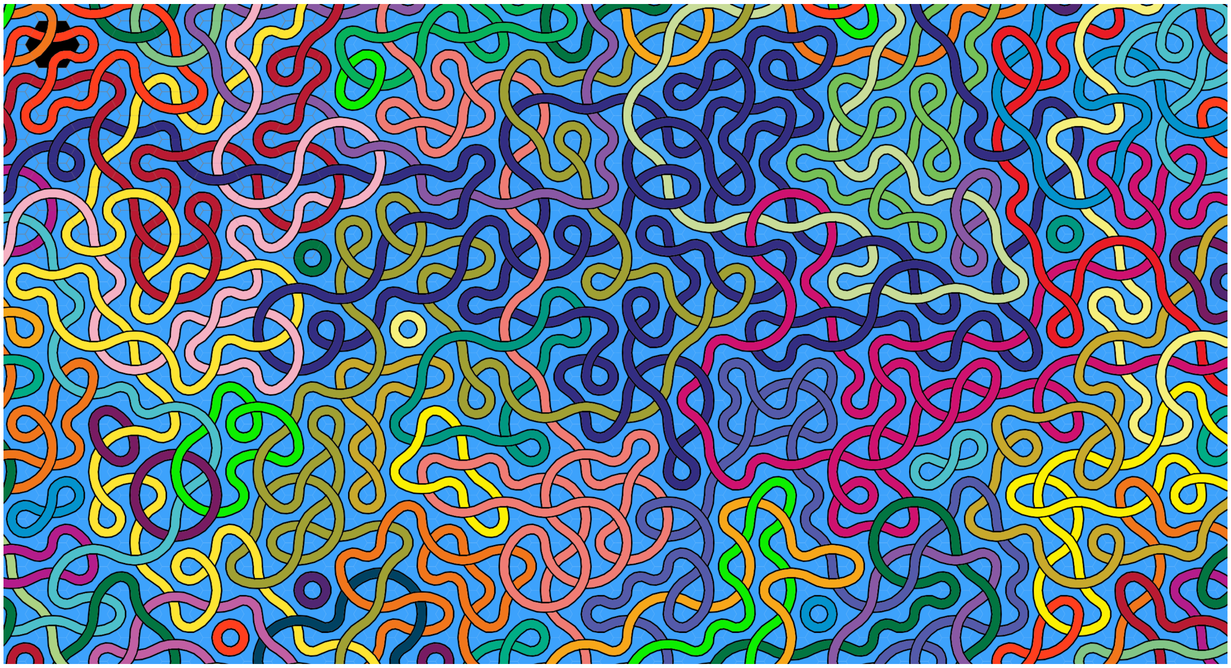

Figure 10: *One simple ribbon hexagon, shown in black at the top left corner, rotated randomly. Individual ribbons are hand-colored to make them stand out.*

Certain edge types are more suitable for ribbon patterns. For example, an AB-CA-BC triangle results in a pattern where only one color connects while the other two colors have dead ends. However, with ribbon patterns, a regular network of rings is formed, as depicted in Figure 11.

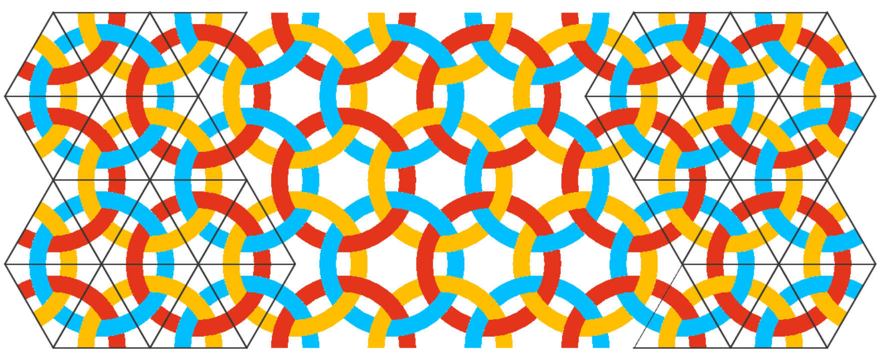

Figure 11: *AB-CA-BC triangles require either the mirror triangle AC-BA-CB or the unmirrored AC-BA-CB. Mirroring affects the ribbon patterns; in this case, the knot changes direction. Observe that the left half of the image uses mirrored triangles while the right half does not. A white background is used to distinguish the ribbons.*

Summary and Conclusions

Simple polygonal tessellations offer immense potential to create various regular and non-repeating patterns. These patterns can be designed, random, or a mix of both. Of course, the level of randomness can be tailored according to the designer's preferences.

I believe these patterns could be effectively utilized in textiles, wallpapers, and ceramic tile applications. The AB-CB hexagon male–female logic could be applied to jigsaw puzzles, and I have several ideas for both board and computer games.

Since the actual Truchet square is a unique case among the multitude of polygons, edges, and patterns, I would not generalize the use of its name to encompass all polygons containing different patterns and forming larger patterns. Instead, "Truchet square" should specifically refer to The Truchet Tile by Father Truchet. A more general term for all polygons containing patterns could be, for example, "patterlygon."

Acknowledgments

I would like to express my gratitude to the Bridges 2020 and 2023 peer reviewers for their insightful and valuable feedback. Furthermore, I want to thank the organizers of the Generative Art [9] and BIODIG [10] conferences for providing me with the opportunity to present my research in the field of patterlygons.

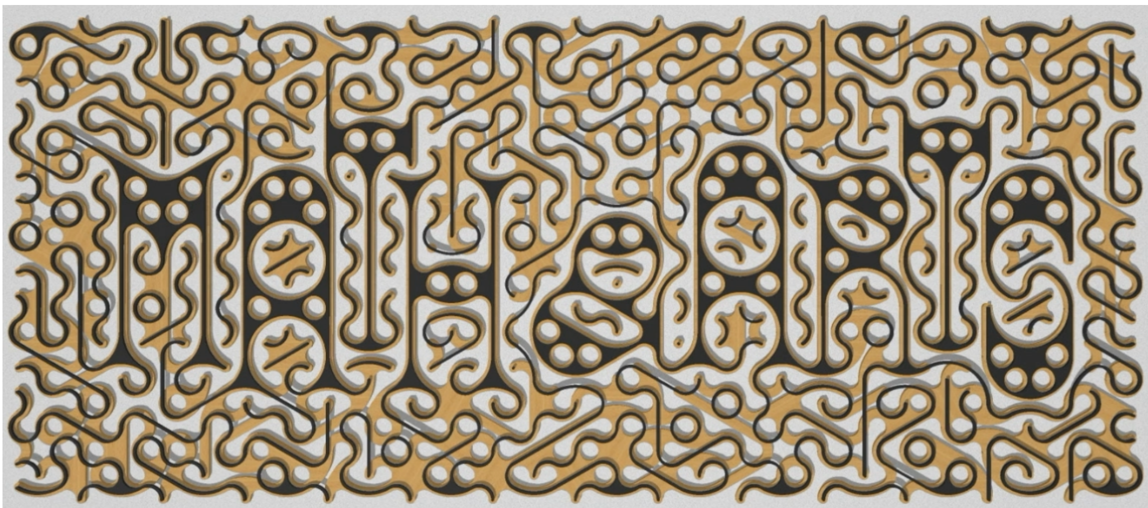

Figure 12: *Example of creating text with layered ABA hexagons*

References

[1] C. Carlson. "Multi-Scale Truchet Patterns." *Bridges Conference Proceedings*, Stockholm, Sweden, Jul. 25–29, 2018. pp. 39–44. http://archive.bridgesmathart.org/2018/bridges2018-39.html

[2] B. Grünbaum and G. C. Shephard. *Tilings and patterns*, W. H. Freeman and Company, 1987.

[3] N. Katz, B. Krietemeyer and T. Schwinn. "Interacting with the Model." In *Inside Smartgeometry: Expanding the Architectural Possibilities of Computational Design*, edited by B. Peters and T. Peters. John Wiley & Sons Ltd, 2013.

[4] K. Mitchell. " Generalizations of Truchet Tiles." *Bridges Conference Proceedings*, Espoo, Finland Aug. 1–5, 2020. pp. 191–198. http://archive.bridgesmathart.org/2020/bridges2020-191.html

[5] G. Pelletier-Auger. *The Tilings of Truchet*. 2017–2018. https://pelletierauger.com/en/projects/the-tilings-of-truchet.html

[6] D. Reimann. "Text from Truchet Tiles." *Bridges Conference Proceedings*, Banff, Canada, Jul. 26–29, 2009. pp. 325–326. http://archive.bridgesmathart.org/2009/bridges2009-325.html

[7] S. Virolainen. *Art for All*, 2019. Wallpaper. Väre building, Aalto University, Espoo. https://youtu.be/3Gr4PxdrBkg

[8] S. Virolainen. "Random Hexagons and Other Patterns Continuities." *XX Generative Art Conference Proceedings*. Ravenna, Italy, Dec. 13–15, 2017. pp. 331–352. http://www.artscience-ebookshop.com/ebooks_free/GA2017_proceedings_ebook.pdf

[9] *Generative Art* https://www.generativeart.com

[10] *BIODIGARQ* http://www.biodigitalarchitecture.com