Sine Waveform Cars

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Abstract

We construct a car with a pen holder which is such that the locus of the pen traces the waveform of the sine curve. Further, we show how adjustments to different aspects of the construction lead to changes in the frequency or the amplitude of the waveform. This is driven entirely by the mechanism that is fitted on the front wheels. By additional tweaks to the rear wheels and by shifting the location of where the pen attaches to the overall mechanism, we also design cars that can "draw" linear combinations of the sine and the cosine functions. By incorporating additional circles to the car, we can get Fourier series, and by adding any number of sinusoids we can approximate any function. A similar machine was made by Michelson almost 100 years ago, although the mechanism for addition of sinusoids and generating sinusoids was completely different.

Introduction

In this paper, we show how to construct a simple car, equipped with a pen holder, with the behavior that if the car is moved along a straight line, the pen draws the waveform of a sine curve. A video of the car in motion drawing various curves can be found at [2][3]. Photographs showing the constructed car are in Figures 1 and 2(a). By adapting the mechanism appropriately, one can change the frequency and the amplitude of the waveform generated. Further, by adding a similar system of gears to the rear wheels and connecting the two, and placing the pen-holder in the middle of the system, the attached pen traces the waveform of not just the sine function, but a linear combination of the sine and cosine functions and other variations depending on the setup. The entire car can be built out of three A4-sized MDF sheets with laser-cut parts as shown in Figure 1. For simple expositions of the history and evolution of trigonometric functions, we refer the reader to [1][4]. To the best of our knowledge, our work as presented here is one of the first illustrations of a clear physical manifestation of the drawing of the sine and cosine curves (and related forms) using a purely mechanical device that is easily assembled and inexpensive to produce.

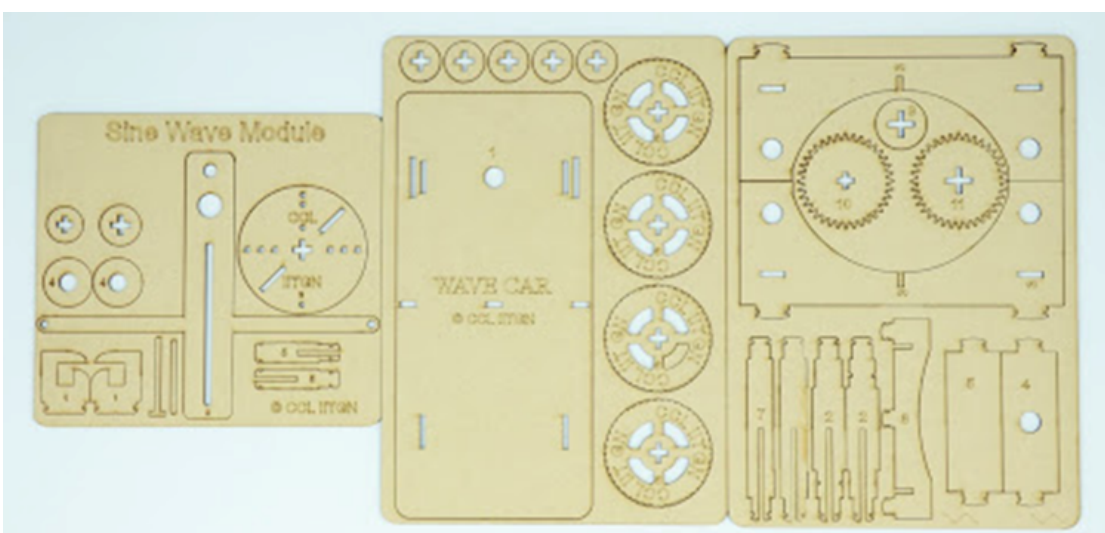
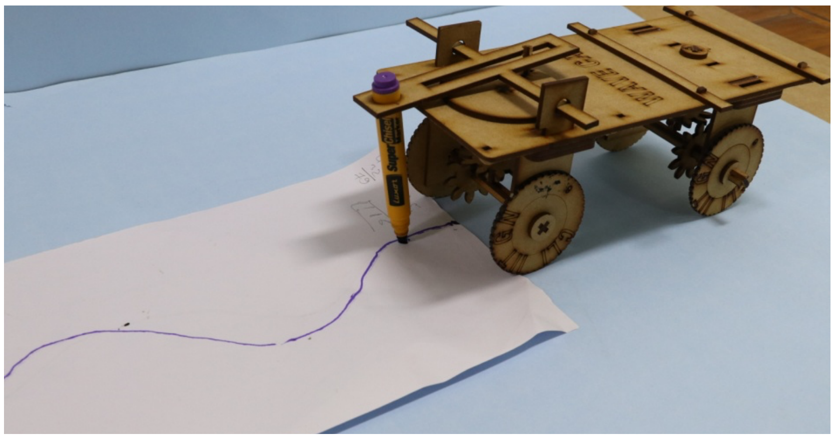


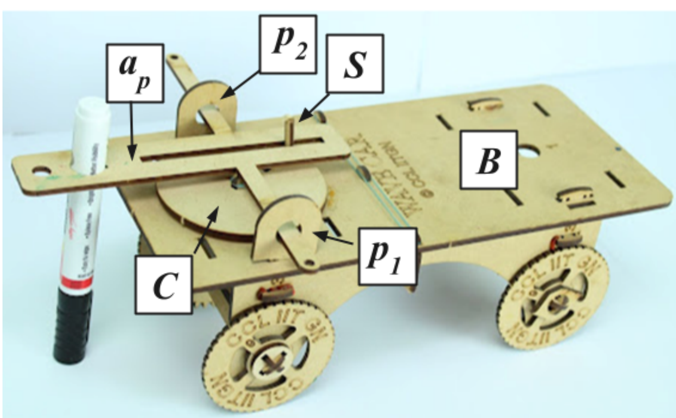
Figure 1: *Design of individual parts of Sine Wave car on MDF sheets.*

The Construction of the Sine Wave Car

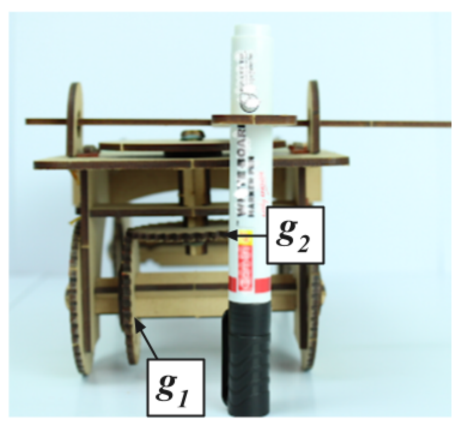
We begin by describing the construction of the car. This description is best read alongside the images in Figure 2(b) and 2(c) depicting the car. To begin with, we have a simple car consisting of a flat base B mounted on four wheels. For now, we assume that the radius of these wheels is fixed and has unit length. We will see later how the radius of the wheels influences the outcome of the path traced by the pen attached to the car. We insert a gear in the middle of the axle joining the front pair of wheels, which we refer to as g_1, with t_1 teeth. Note that the plane of this gear is parallel to the wheels, and is therefore perpendicular to the ground. We attach to this gear another one, g_2, with $t_2$teeth that is perpendicular to g_1. Note that g_2 is parallel to the ground. In the basic system, we will have $t_1 = t_{2,}$. We attach a circular disc C with five holes to g_2 using a rod (see Figure 2(b)), and this disk is mounted on top of the base. We have a small stick S that can fit into any of the holes, and once in place, rests perpendicular to the base.



(a)



(b)



(c)

Figure 2: *(a) Sine wave car after construction; (b) and (c) Annotated pictures of Sine wave car.*

The pen holder is a simple attachment, which we refer to as a_p, shaped as shown in Figure 3(b). It is mounted on top of the disk and is held in place by the plates p_1 and p_2 as can be seen from the Figure 2(b). Observe that the plates p_1 and p_2 do not restrict the movement of the plate in the horizontal direction, which is perpendicular to the motion of the car. Also note that S, which is the stick, is driven in a circular motion as the wheels move, and therefore this drives the attachment a_p accordingly.

The Path Traced

We now turn to a discussion of the path that is traced by the pen that is placed inside the circular hole of the attachment a_p. For convenience, we will measure the angles in radians during this discussion. Note that the pen is perpendicular to the ground. On a practical note, we remark that the pen must be attached in such way that its tip touches the ground firmly, and that it fits well into the space in the attachment a_p. First, imagine placing the car so that the tip of the pen is at the origin (0, 0) of a Cartesian coordinate system. Now let us move the car forward in the direction of the x-axis, and assume that the wheel has rotated θ radians. The tip of the pen is now at a location (x_θ, y_θ). First, observe that x_θ is equal to the length of the arc of the circle subtended by an angle of θ. Indeed, this is simply the distance that the car has moved along the x-axis when the wheel has rotated by θ which is given by $x_\theta = r\theta = \theta$, where r is the radius of the wheels.

The second equality follows because we assumed the wheels of our car to have unit length radii. As a special case, note that with one full turn of the wheel, the car would have moved forward 2π units, which is exactly the circumference of a circle with unit radius. The key issue is determining y_θ. When the wheel rotates θ radians (say in the clockwise direction for forward motion), g_1 rotates θ radians in the clockwise direction, while g_2 rotates θ radians in the anticlockwise direction (this is because the two gears have the same number of teeth in our setup and are interlocking). This causes the circular disc C to also move θ radians in the anticlockwise direction. The stick attached to the disc also moves accordingly, pushing the attachment a_p, and the pen attached, a certain distance along with it. This distance, we claim, is given precisely by $\sin(\theta)$ (assuming the radius of the circle on disc C is 1) and this can be understood from Figure 3(a).

Modifying the Amplitude and Frequency

Let us now describe how we can extend this mechanism to change the frequency or the amplitude of the waveform generated by the car. The highest point (crest) of the waveform is the radius of the circle on the top of the car as shown in Figure 3(a). The distance of the slots of this circle defines the farthest the pen moves and so define the amplitude of the wave. So, the distance from the centre of the circle to the slot in which we place the stick becomes the amplitude of the waveform generated. When we fix the stick in the farthest slot on the circle, we get the waveform with the largest amplitude. It is considered to be $\sin(\theta)$, i.e. the waveform with circle of radius 1. To change the frequency of the sine wave, we can change the ratio of the wheel and circle assembly. So, if we change the gear ratio such that for one turn of the wheel the circular base turns twice, the frequency of the sine curve doubles. In other words, now the car draws the curve $y = \sin(2\theta)$. This gear assembly can be done by reducing the number of teeth t_2 in g_2 or increasing the number of teeth t_1 in g_1 to achieve a 2:1 gear ratio.

(a) (b)

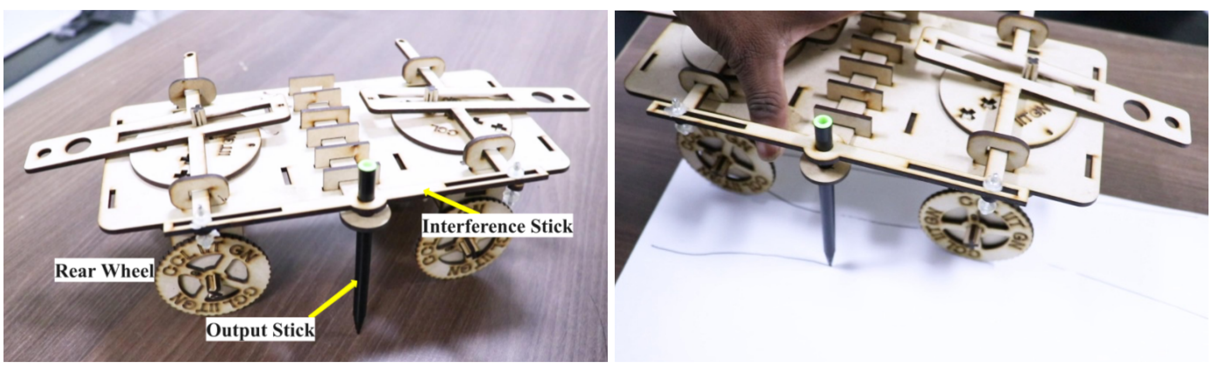
Figure 3: *(a) The circular base C that rests just below the attachment and is connected to the base B (b) The pen holder a_p which is mounted on top of the base and is therefore parallel to the ground.*

Linear Combinations: From Sine Wave Nano to Fourier Ferrari

There are two sets of wheels in the car and if we replicate the mechanism on the first wheel on the rear wheel, we get a car which can draw two sine waves independently and simultaneously (see Figure 4). Now instead of the penholder we have sticks which will draw the two sine waves. We can attach another stick on top of these two sticks and have a pen attached on this output stick such that the pen always moves in a straight line perpendicular to the car. If the pen is attached on this interference stick in the middle, the output will be average of two sine waves. The position of output stick (on the interference stick) can be adjusted to get the weighted sum of the two sine waveforms. We have also designed different wave modules which can fit on the car and can draw a triangular wave, cycloid, trochoids, cardioid, etc. When the length of the car is not a constraint, we can add several wave modules with different frequencies and add them using the interference mechanism thereby making a Fourier Series car. So, if we have 4 circular discs spinning with frequency 1×, 2×, 3× and 4× of that of the wheel, then the sticks connected to these discs will draw $\sin(\theta)$, $\sin(2\theta)$, $\sin(3\theta)$, $\sin(4\theta)$. We can then, for instance, join the first two sinusoids by interference stick and next two by another stick. And then join these two interference sticks with another interference stick and so the final output will be $\frac{\sin(\theta)+\sin(2\theta)+\sin(3\theta)+\sin(4\theta)}{4}$. The amplitude of each of the sinusoids can be changed by adjusting slots in the circular base of respective sinusoid. We can thus convert our Sine Car to a Fourier Ferrari.

Summary and Conclusions

We believe that this car could be a useful addition in classroom settings to help students and teachers develop a feel for trigonometric and algebraic functions, their combinations, and beyond. The car can be easily made and assembled and that experience itself is a great inspiration for the student. With just A4 size MDF's and a laser cutting facility, we have produced several (about 100) hands-on experiential activities which can lead to better conceptual understanding of several key ideas in physics and mathematics.



(a) (b)

Figure 4: *(a) Car with two sine modules and their linear combination using interference stick (b) Trace of the pen showing interference of two sine waves*

References

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[3] M. Jain. *Video Demonstration of Interference Car.* https://youtu.be/B4FurVb1PlM

[4] E. Maor, *Trigonometric Delights (Princeton Science Library)*. Princeton University Press, 2013.