An Integer Square Variant of the Harriss Spiral

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Abstract

Edmund Harriss developed a branching spiral based on a decomposition of a rectangle whose sides are in the plastic ratio into a square and two smaller rectangles. This concept was inspired by the golden spiral, a classical self-similar figure induced by decomposing a rectangle into a square and a similar rectangle. Neither of these decompositions can be realized in a medium where the side lengths of individual components must be integers, but the golden spiral's discrete variant, the Fibonacci spiral, can be so realized. This work proposes and analyzes an integer-side-length variant of the Harriss spiral.

The Golden and Fibonacci Spirals

One of the widely celebrated aesthetic properties of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ is that a rectangle whose side lengths are in the golden ratio can be decomposed into a square and a smaller rectangle in the same ratio. The smaller rectangle can itself be decomposed into a smaller square and yet smaller rectangle, and so on indefinitely; the resulting arrangement, if one consistently rotates the direction of the decomposition by 90 degrees in the same direction, is a spiral of smaller and smaller squares. This process appears in Figure 1(a). A quarter circle is often drawn inside each square, as seen in this figure, to accentuate the spiraling nature of the squares.

This decomposition of a rectangle into several squares in different sizes is appealing for such arts as crochet or quilting, where the stitching together of separate geometric elements, especially squares, is an established form. However, specifically for the purposes of representing this design in crochet, the fact that the golden ratio is irrational is a significant liability. While in quilting, fabric can be cut to any length desired, crochet requires an integer number of stitches. In all but the lightest of yarns or in thread crochet, the sizes of individual stitches would make it likely that simply rounding square sizes to the nearest integer would, in some places, lead to squares which are not the right size to be joined together.

The golden spiral, however, has a variant, the *Fibonacci spiral*, in which all lengths are integers, and are specifically the Fibonacci numbers defined by the recurrence $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Just as the golden spiral is produced by decomposing a $1 \times \phi$ square into a 1×1 square and the similar $\phi^{-1} \times 1$ rectangle, so does the Fibonacci spiral decompose a $F_{n+1} \times F_n$ rectangle into a $F_n \times F_n$ square and a $F_{n-1} \times F_n$ rectangle. Notably, in the Fibonacci spiral the smaller rectangle is not similar to the original rectangle but is decomposable in the same manner. Whereas the golden spiral is infinite, a Fibonacci spiral is of necessity finite in its descent, since it will eventually reach a decomposition into 1×1 rectangles. A

(a) *The golden spiral* **(b)** 13×8 *Fibonacci spiral*

Figure 1: *The golden spiral and its integer approximation*

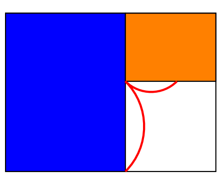
Figure 2: *A Fibonacci spiral of seven granny squares*

Fibonacci spiral decomposed into 6 squares appears in Figure 1(b); it is not identical to the golden spiral but, if the original rectangle starts with high enough Fibonacci numbers, it appears very similar. This similarity arises from a close correspondence between the golden ratio and the Fibonacci numbers, specifically that the limit of the ratio of consecutive Fibonacci numbers approaches the golden ratio.

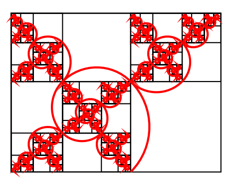
Because its side lengths are integers, a Fibonacci spiral is amenable to being stitched from crocheted granny squares, and the blanket in Figure 2 uses this property to present a near approximation of the golden spiral.

The Harriss Spiral

Edmund Harriss designed a variant of the decomposition for the golden spiral in which a rectangle is decomposed into three smaller units: a rectangle similar to the original rotated 90°, a square, and a similar rectangle in the same orientation as the original rectangle [1][3]. This decomposition appears in Figure 3(a). As in the golden-spiral decomposition, the individual non-square units can be decomposed further along these lines to create a cascading filling of the rectangle with ever-smaller squares. Unlike in the golden spiral, however, each square is incident on two smaller regions appearing in the same generation; if arcs are drawn between each square and the square which appeared in its previous generation, we thus get a branching structure, shown in Figure 3(b).



(a) *The Harriss decomposition, with arcs*



(b) *Iterated Harriss decomposition*

Figure 3: *The construction of the Harriss spiral*

As with the golden spiral decomposition, the Harriss spiral requires a specific aspect ratio for the original rectangle. While the golden spiral requires an aspect ratio which is a solution to $\phi^2 = \phi + 1$, the Harriss spiral requires an aspect ratio ρ satisfying $\rho^3 = \rho + 1$, whose real solution is known as the *plastic ratio* and equals

$$\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} \approx 1.3247.$$

Like the golden ratio, this quantity is unsuitable for integer-length crafting techniques such as crochet, which motivated a search for a Fibonacci-spiral analogue to the Harriss decomposition.

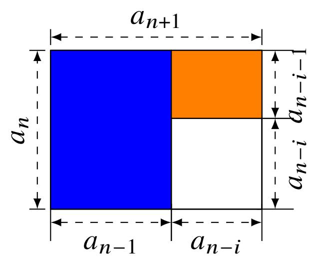


Figure 4: *An integer-sequence decomposition akin to the Harriss design*

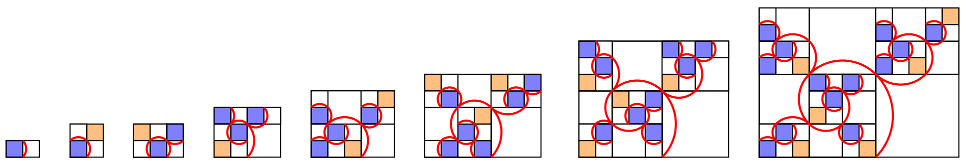


Figure 5: *Branching spirals based on the Padovan sequence*

Padovan Numbers

Just as the Fibonacci spiral relaxed the rectangle similarity condition for the golden spiral, it is possible to relax the similarity condition on the Harriss spiral. Instead of demanding that the rectangles all be in the same aspect ratio, we may establish that, for some sequence $a_0, a_1, a_2, \ldots$, a rectangle is a valid one to use either as the original rectangle in an integer-Harriss decomposition, or as any of the nonsquare rectangles in an integer-Harriss decomposition, if it has dimensions $a_n \times a_{n+1}$ (or a 90° rotation thereof) for some n.

Since our decomposition should utilize the entire sequence, and since the original rectangle and its largest subrectangle share an edge, we shall assume that, if the original rectangle has size $a_{n+1} \times a_n$, its largest subrectangle should have size $a_{n-1} \times a_n$. It is possible that the smaller rectangle comes from considerably earlier in the sequence, so we establish its dimensions to be $a_{n-i} \times a_{n-i-1}$ for some $i \geq 1$, and then the square has sides both of length a_{n-i}. This schematic appears in Figure 4. In order for all the edges to line up correctly and fill the original rectangle, it thus follows that:

$$a_n = a_{n-i-1} + a_{n-i}$$
$$a_{n+1} = a_{n-1} + a_{n-i}$$

Substituting $n+1$ in for n in the first equation and equating it to the second yields $a_{n-i+1} = a_{n-1}$, so in order for this sequence to be nontrivial we must have $i = 2$, and the governing equation of this number sequence is $a_n = a_{n-2} + a_{n-3}$. Several sequences satisfy this recurrence, but since the eventual goal is a reduction to 1×1 squares, we may establish $a_0 = a_1 = a_2 = 1$ as initial conditions. The resulting sequence is of the *Padovan numbers*, with OEIS reference number A000931 [2].

The resulting decomposition could be called the *rectangular Padovan spiral* by analogy to the Fibonacci spiral. The procedure to build such a spiral is to start with a rectangle whose side lengths are two consecutive Padovan numbers, and then to repeatedly decompose each $a_{n+1} \times a_n$ rectangle into an $a_{n-1} \times a_n$ rectangle, an $a_{n-2} \times a_{n-2}$ square, and an $a_{n-2} \times a_{n-3}$ rectangle until all regions are square. This process is shown for several initial rectangle sizes in Figure 5. For very small starting rectangles, the result is unsatisfactory, but a structure akin to the Harriss spiral eventually emerges. As in the case of the Fibonacci spiral, the limiting aspect ratios of the two spirals are the same. Just as the ratios of consecutive Fibonacci numbers converge to the golden ratio, the ratios of consecutive Padovan numbers converge to the plastic ratio.

Table 1: *The number of each component in Padovan-spiral decompositions.*

Rectangle size	1×1					2×2		3×3	4×4
	blue	orange	$a_0 \times a_0$	$a_1 \times a_1$	$a_2 \times a_2$	$a_3 \times a_3$	$a_4 \times a_4$		
2×1	1	0	1	0	0	0	0	0	0
2×2	1	1	1	1	0	0	0	0	0
3×2	2	1	1	1	1	0	0	0	0
4×3	3	1	2	1	1	1	0	0	0
5×4	4	2	3	2	1	1	1	0	0
7×5	6	3	4	3	2	1	1	1	0
9×7	9	4	6	4	3	2	1	1	1

Constituent Elements in the Rectangular Padovan Spiral

The Padovan spiral decomposes the entire rectangle into several squares, many of which are 1×1. Those 1×1 squares themselves can be thought of as being in several families, even though they are geometrically identical: those that arise as large rectangles in the decomposition (colored blue throughout this paper), those that arise as squares in their own right (colored white), and those which are small rectangles (orange). The several possible constituent elements of Padovan spiral decompositions are listed in Table 1 for a few initial rectangle sizes, with a specific distinction made among several different ways 1×1 and 2×2 white squares can show up, since the numbers 1 and 2 both appear multiple times in the Padovan sequence. In this table, each column, after the first appearance of a 1, satisfies the recurrence $b_n = b_{n-1} + b_{n-3}$; this occurs because the $a_{n+1} \times a_n$ rectangle decomposes into a single $a_{n-2} \times a_{n-2}$ square together with all the constituent components of a $a_n \times a_{n-1}$ rectangle and all the constituent components of a $a_{n-2} \times a_{n-3}$ rectangle. With the appropriate categorization of the 1×1 and 2×2 squares by which term of the Padovan sequence is used to create them, the number of components of each type are all offsets of the exact same sequence, the Naryana's cows sequence, with OEIS reference number A000931 [2]. The total numbers of 1×1 squares and 2×2 squares are sums of several offsets of this sequence, such that total the number of 1×1 squares is simply twice the Naryana's cows sequence, and the number of 2×2 squares is the sequence with OEIS reference number A097333 [2].

Future Work

The granny-square afghan based on these principles is yet to be constructed, but a Padovan-spiral variant of the work in Figure 2 is in progress. In addition, the ideas developed here are applicable to other self-similar decompositions of rectangles. Harriss has described such decompositions in general as *proportion systems* [1], and although many proportion systems demand irrational aspect ratios, the same recurrence-based description developed here should be applicable to produce integer analogues of each of them.

References

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[3] M. Scroggs and E. Harriss. "On the cover: Harriss Spiral." *Chalkdust*, no. 9, 2019. https://chalkdustmagazine.com/regulars/on-the-cover/on-the-cover-harriss-spiral/.