Whirling Tiles

Ulrich Reitebuch[1], Martin Skrodzki[2], Henriette Lipschütz[3], Konrad Polthier[4]

[1]Institut für Mathematik, Freie Universität, Berlin, Germany; Ulrich.Reitebuch@fu-berlin.de
[2]Computer Graphics and Visualization, TU Delft, Netherlands; mail@ms-math-computer.science
[3]Institut für Mathematik, Freie Universität, Berlin, Germany; Henriette.Lipschuetz@fu-berlin.de
[4]Institut für Mathematik, Freie Universität, Berlin, Germany; Konrad.Polthier@fu-berlin.de

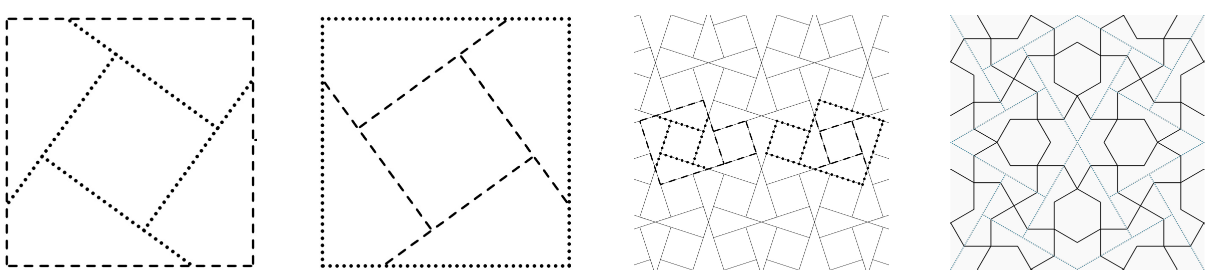
Abstract

Whirling kites, a pattern occurring in Islamic geometric ornaments, is the main inspiration for this paper. First, the pattern is presented as well as different perspectives on how to create it. Then, one of these perspectives is generalized in order to obtain a family of patterns. Finally, we apply the generalization to show these patterns on several tilings.

Whirling Kites

Islamic geometric ornaments have both a rich mathematical background and an artistic quality to them. As such, they have been investigated within the Bridges community [2][5][6]. In this paper, we will consider a specific Islamic geometric pattern, coined *whirling kites* by Peter R. Cromwell and Elisabetta Beltrami [4].

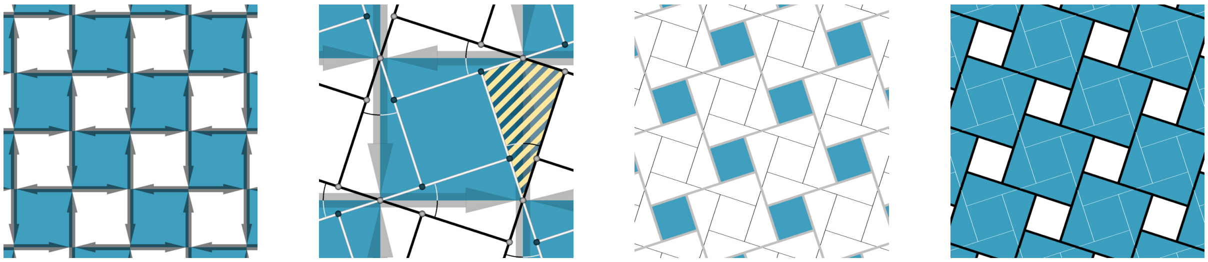
The whirling kites pattern is constructed from congruent kites, which are cyclically arranged around a small central square in such a way that the four kites together with this center square form a bigger square (Figure 1(a)). This pattern can be used as a decoration for a single square, as illustrated, e.g., in Figures 12 and 14 of [4], which show photographs from the Friday Mosque in Isfahan. However, the pattern can also be used to tile the plane periodically (Figure 1(c)), which will be the more interesting aspect for this paper.



(a) *Whirling kites single square, clockwise.* (b) *Whirling kites, counter-clockwise.* (c) *Periodic arrangement of whirling kites.* (d) *Pattern from the Topkapı scroll.*

Figure 1: *The whirling kites pattern in its two orientations on a single square and in a periodic arrangement as well as an application example from the Topkapı scroll.*

There are two mirror symmetric ways to arrange the four kites around the small square; compare Figures 1(a) and 1(b). Note that in the periodic pattern both are present (Figure 1(c)). The periodic pattern can be interpreted as a pattern of large and small squares, highlighted dashed or dotted in Figure 1(c). In this point of view, the large squares are cut into four kites and a small square each. Then, the new small squares at the center of the large ones are drawn at a different angle than the "original" small squares. Around each "original" small square, there is now an arrangement of four whirling kites, which are whirling in the opposite orientation as those that make one of the "original" large squares. Thus, each kite can be seen as part of two large squares and it can whirl in one or the other orientation around a small square. Ornaments utilizing this pattern can be found, e.g., in the Topkapı scroll [7] (late 15th/early 16th century). We reproduced an example from the scroll in Figure 1(d) with the whirling kites pattern dotted in the background.



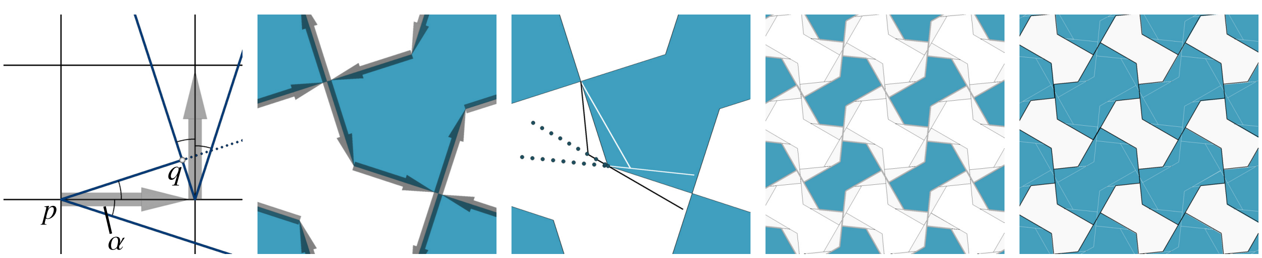
(a) *Checkerboard with oriented edges, cyclic around each square.* (b) *Splitting each square into four right-angle triangles and a square.* (c) *Light coloring of the kites gives clockwise whirling kites.* (d) *Dark coloring gives counter-clockwise whirling kites.*

Figure 2: *Construction of the whirling kites pattern from a square tiling.*

A different construction starts from a regular square grid. As the squares in the grid are two-colorable, we can impose an orientation on the edges between two squares that is cyclic around each square (Figure 2(a)). Next, we cut each square into four pairwise congruent right triangles and a small square, analogous to a visual proof for the theorem of Pythagoras (Figure 2(b)). This is done alternatingly in both possible orientations, indicated by the oriented edges, i.e., alternating in the light and dark squares. As a result of the construction, the right triangles on both sides of each oriented edge it can be composed into a kite (shaded area in Figure 2(b)). Coloring the kites light (Figure 2(c)) or dark (Figure 2(d)) highlights the two different orientations of the small squares, as observed in the periodic arrangement of Figure 1(c). We proceed to generalize this construction.

Generalizing the Construction

In this section, we apply a similar construction to other input tilings, more general than just square tilings. A necessary condition on the input tiling is that it has to be two-colorable, i.e., there are only two colors needed to ensure that tiles having edges in common have different colors. Thereby, as in the square grid, we can give all edges an orientation in such a way that the edges around tiles of one color are all clockwise, while those around tiles of the other color are all running counter-clockwise.



(a) *Line construction.* (b) *Oriented tiling.* (c) *Both directions.* (d) *Light coloring.* (e) *Dark coloring.*

Figure 3: *Generalizing the whirling kites construction to cyclic quadrilaterals on two-colorable tilings.*

On such an oriented input, we perform the following construction. On each oriented edge, at the starting point p of the arrow, construct two straight lines intersecting at the start point, with a fixed angle α to the edge (Figure 3(a)), where the admissible range of α depends on the specific tiles as large α can degenerate the construction. Then, the edge becomes the angular bisector of the angle between these two lines. Extend all these lines simultaneously and stop each line at the first intersection point q with a line originating at the next edge in both the edge's adjacent tiles. These intersection points and the edge endpoints constitute a quadrilateral, which on a square tile input is exactly the kite described above.

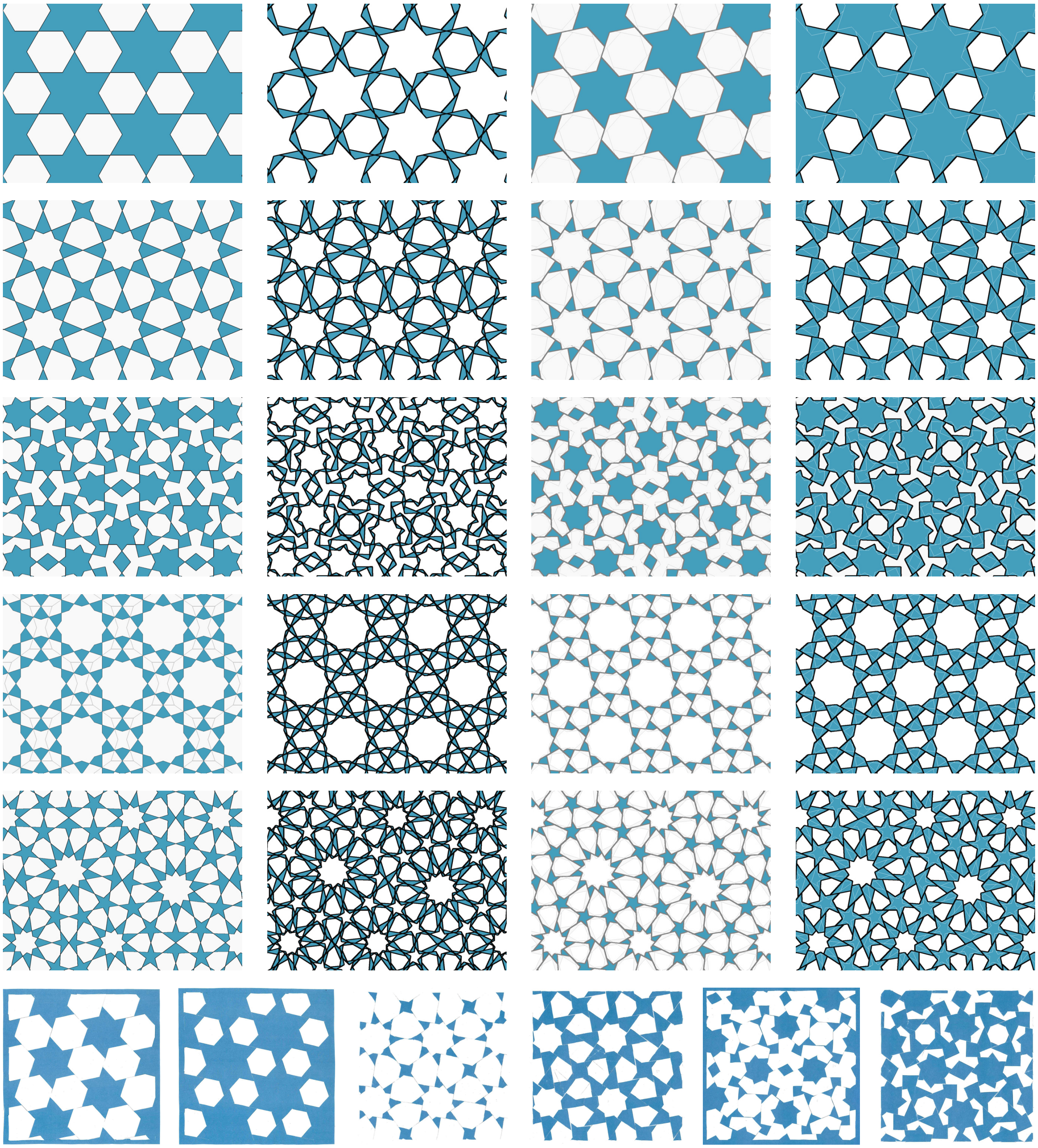


Figure 4: *Top to bottom: tilings from Bourgoin panels 47, 88, 216, and 221 (see [3]), an almost regular tiling from the Topkapı scroll, panel 42 (see [7]). Each row shows from left to right: original tiling, constructed cyclic quadrilaterals, light coloring, and dark coloring. Last row: The resulting tilings for Bourgoin panels 47, 88, and 216 were paper crafted similar to [8].*

If the start of the arrow is a point of degree four, i.e., a point where four tiles meet, it suffices to draw the lines in the direction of the arrow. We take this generalization one step further. To illustrate that, consider the Cairo pentagonal tiling, with pairs of pentagons glued together to obtain a tiling with octagonal tiles. This tiling can then be oriented as described above (Figure 3(b)). For points of degree four, we perform the same operation as outlined above and illustrated in Figure 3(a). However, now there are corner points of degree two. For these, we simply extend the two lines also in the backwards direction of the arrow, starting from the base of the arrow (Figure 3(c)). Thereby, we can also apply the construction if corner points are present.

Observe that if the points of the input have two or four adjacent edges—in the latter case two pairs of collinear edges with point symmetric directions—the constructed quadrilaterals are not necessarily kites, but are always cyclic quadrilaterals. This is since the angles at the end of each edge sum up to π. By the point symmetric edge directions in the input at points of degree four, the constructed cyclic quadrilaterals also will have aligned edges and can be composed to big tiles without corners at input points (Figures 3(d) and 3(e)).

Examples and further Observations

In this section, we will apply the construction outlined above to several tilings from [3] and [7] and present the results. Note that the resulting patterns can be regarded from different perspectives, just as the original whirling kites have a clockwise and a counter-clockwise orientation. In any case, after constructing the cyclic quadrilaterals, there are small remaining tiles of the tiling, where the edges are alternatingly rotated in clockwise and counter-clockwise direction by the input fixed angle (in all our examples, we use $\alpha = 12°$). Depending on how to color the cyclic quadrilaterals, light or dark, the construction gives rise to composed large light tiles and small dark tiles rotated in counter-clockwise direction or composed large dark tiles and small light tiles rotated in clockwise direction, respectively. Note that all these tiles have the same angles as the input tiles of the original tiling. However, while the tiles were similar for the original whirling kites (Figure 1(c)), they are not for the generalized construction. Here, the change of edge lengths depends on the edge length of the input tiles and the inner angles, where each row shows, from left to right: original tiling, constructed cyclic quadrilaterals, light coloring, and dark coloring, as illustrated in Figure 4.

References

[1] C. Bier. "Islamic Art: An Exploration of Patterns." *Bridges Conference Proceedings*, London, UK, Aug. 4-8, 2006, pp. 525–532. https://archive.bridgesmathart.org/2006/bridges2006-525.html

[2] J. Bonner. "Three traditions of self-similarity in fourteenth and fifteenth century Islamic geometric ornament." *Bridges Conference Proceedings*, Granada, Spain, Jul. 23–26, 2003, pp. 1–12. https://archive.bridgesmathart.org/2003/bridges2003-1.html.

[3] J. Bourgoin. *Les éléments de l'art arabe*, Firmin-Didot et cie, 1879.

[4] P. R. Cromwell and E. Beltrami. "The whirling kites of Isfahan: geometric variations on a theme." *Math. Intelligencer*, vol. 33, no. 3, 2011, pp. 84–93.

[5] C. S. Kaplan. "Computer generated Islamic star patterns." *Bridges Conference Proceedings*, Winfield, USA, Jul. 28–30, 2000, pp. 105–112. https://archive.bridgesmathart.org/2000/bridges2000-105.html.

[6] M. Kharazmi and R. Saranghi. "Geometric Study of Architectural Designs on a Twelfth Century Structure." *Bridges Conference Proceedings*, 2014, Seoul, Korea, Aug. 14–19, pp. 175–182. http://archive.bridgesmathart.org/2014/bridges2014-175.html.

[7] G. Necipoğlu. *The Topkapı Scroll: Geometry and Ornament in Islamic Architecture*, Getty Center for the History of Art and the Humanities, 1995.

[8] U. Reitebuch, H.-S. Lipschütz, and K. Polthier. "Weaving Paper into Star Patterns." *Bridges Conference Proceedings*, 2020, pp. 427–430. http://archive.bridgesmathart.org/2020/bridges2020-427.html.