Developing a Sculpture of the Trihelical Square Tiling

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Abstract

The trihelical square tiling (TST) is a three-dimensional lattice with unique geometric properties and aesthetic appeal. I discuss some such properties, recount my process of realizing the TST as a modular sculpture, and consider directions to take future artwork inspired by the TST.

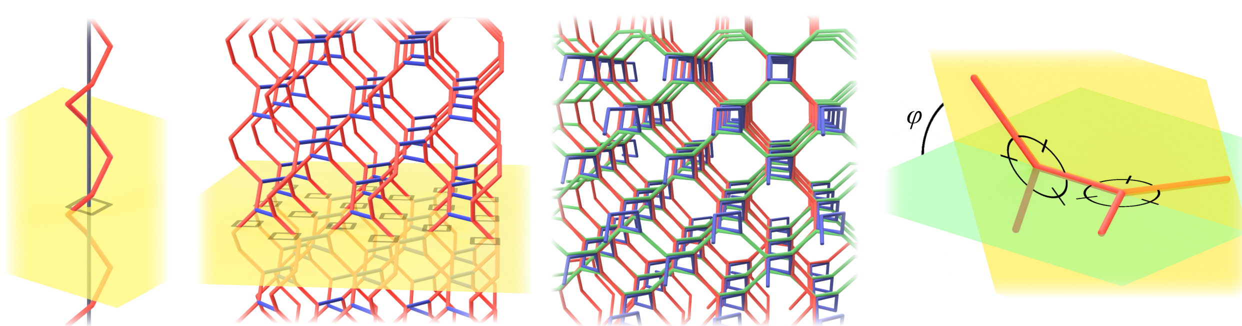
Introduction

A certain three-dimensional lattice has intrigued mathematicians, chemists, physicists, and artists for nearly a century. Due to many independent discoveries, it holds many names, including the Laves graph [2], the $(10, 3)$-a network [4], the Triamond net [1], and the K_4 crystal [9], but I will use the name I first learned: the *trihelical square tiling* [3], or TST. The TST lattice is triply periodic, chiral, and vertex- and edge-transitive. It is intimately related to the gyroid minimal surface and certain crystal structures, along with other natural and geometric phenomena [5].

The TST has been analyzed extensively across a variety of disciplines. So in this paper, I will focus on the engineering challenges and aesthetic considerations relevant to my construction of a physical model of the TST. I will begin by defining the TST geometrically, then discuss the development of my sculpture, and conclude with goals for future artistic endeavors.

Geometry of the TST

The TST lattice can be defined in a few equivalent ways; what follows is the definition I first learned [3]. We begin by constructing an infinitely long square helix about vertical axis Z, with edges of length 1, where each edge has a 45° slope w.r.t. horizontal plane XY (Figure 1(a)). The helix is "square" in that we incrementally rotate each edge 90° about Z, meaning its projection onto XY is a square.



(a) **(b)** **(c)** **(d)**

Figure 1: *Construction process of the TST lattice, and visualization of its properties. (b) and (c) use a perspective view, rather than orthographic, for clarity regarding depth.*

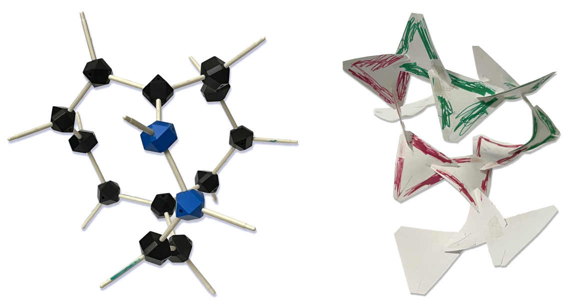
If we infinitely "step and repeat" this helix over XY with just the right spacing, we can connect adjacent helices using edges of length 1 which are parallel to XY; these edges are shown in blue in Figure 1(b). Doing this has a surprising result: there are now two more tilings of square helices about the YZ and ZX planes, both congruent to the initial XY tiling. These three tilings are shown in red, green, and blue in Figure 1(c). Thus, we have constructed the trihelical square tiling. Interactive 3D views of the TST are available at [8].

Curiously, not only is the TST composed of three sets of square helices, it is equivalently composed of four sets of triangular helices [5]. The axes of these square and triangular helices correspond to the normals of the square and triangular faces of a cuboctahedron.

We can easily derive some key properties of the TST, as summarized in Figure 1(d). Every vertex v has exactly three adjacent neighbors, which are equiangular about and coplanar with v. The latter means we can associate a plane with each vertex. The angle between the planes of any two adjacent vertices is $\arccos \frac{1}{3} \approx 70.5°$ (equal to the dihedral angle of a regular tetrahedron). We will simply call this angle φ.

Building a Physical Model

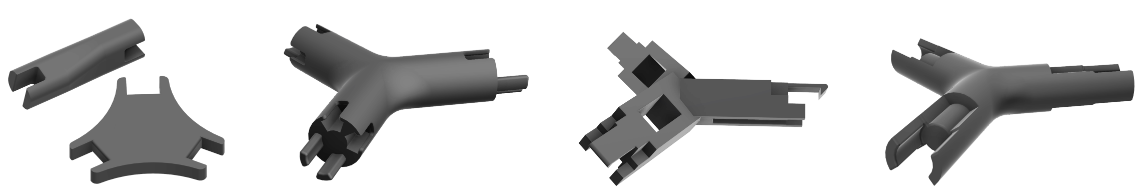
Enamored with the geometry of the TST, I sought to build a tangible model, hoping to deepen my intuition for its structure. Some proofs of concept using materials on hand (Figure 2) made clear that assembling the TST's vertices and edges into proper alignment is difficult when the φ angle is not physically constrained.



(a) **(b)**

Figure 2: *Early tests of a physical TST model using (a) an organic chemistry kit and (b) paper triangles.*

Unsatisfied with these tests, I pursued creating a modular, 3D-printable building system. Assembling many small modular pieces is generally easier than printing a large 3D structure in place, full of voids and overhangs — at least, this is the case with FDM printing, which I had at my disposal. Other manufacturing techniques are more amenable to complex in-place designs; George Hart's interpretations of the TST created with SLS 3D printing demonstrate this well [4].



(a) **(b)** **(c)** **(d)**

Figure 3: *CAD models of each modular TST piece prototype, with (d) being the final design.*

The chemistry set (Figure 2(a)) informed my first attempt at a modular system, its only flaw being the free rotation of vertices/atoms about their edges/bonds. I modeled a vertex piece and an edge piece, where each vertex featured 3 joint locations, and each edge connected two vertices rigidly at the appropriate angle (Figure 3(a)). This first prototype was promising. However, I realized from my paper model (Figure 2(b)) that the edges and vertices needn't be separate pieces. Instead, each piece could constitute one vertex and three half-edges; I just needed to design a joint that would enforce φ when joined *with itself*. The Verhoeffs' *Bamboozle* sculpture leverages this exact idea [10], the pieces of which indeed resemble the slotted triangles of Figure 2(b).

My next two prototypes played with this self-joining piece concept (Figures 3(b) and 3(c)), but both versions suffered from small, fragile details that were difficult to print. Eventually, I created a design that worked well (Figure 3(d)); it was simple to print, with tolerances easy to adjust until the pieces could press-fit together like a typical building toy. Figure 4 clarifies how two of these pieces join to one another.

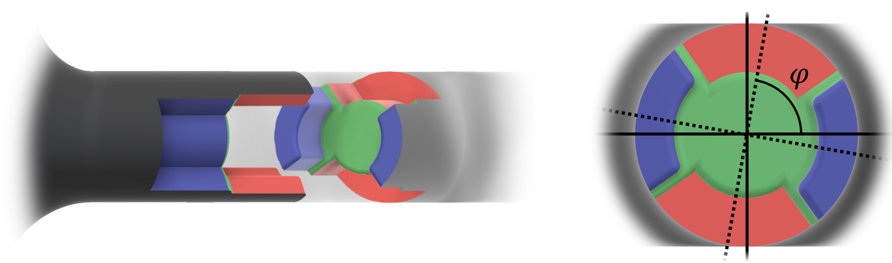


Figure 4: *Two adjoining pieces (left), and the rotation of a joint w.r.t. the plane of its piece (right).*

Pleased with this design, I printed many pieces out of PLA plastic and assembled them into a model of the TST. A benefit of the symmetric style of joint, along with the intrinsic symmetries of the TST, is that no two pieces can ever be connected “backwards.” In other words, as more and more pieces are joined together, they are guaranteed to build out the TST lattice correctly.

Results and Future Work

Two years after completing my model, I was afforded the opportunity to display it at the Joint Mathematics Meetings 2023 Art Exhibition [7]. To prepare, I restructured the outer boundary of the model to be more aesthetically pleasing, and added rounded end caps to the outermost edges. I also introduced plastic straws to represent the axes of the square and triangular helices. The final result is shown in Figure 5.

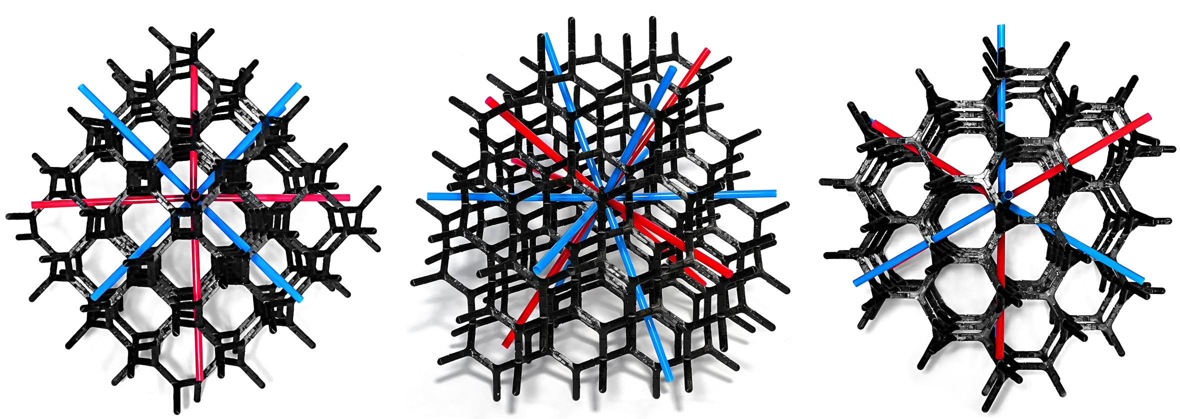


Figure 5: *The final sculpture,* Trihelical Square Tiling*, as presented at JMM'23. The left and right perspectives look “down” the square and triangular axes respectively.*

To test the success of my modular system as a building toy, I provided some spare pieces beside the sculpture at the gallery, and invited patrons to connect them together freely without instruction. Those who played with the pieces seemed to find connecting them intuitive and enjoyable.

Having completed my prototype model, I intend to create more works that convey compelling properties of the TST. Namely, I hope to design a sculpture that truly emphasizes the TST's square and triangular helices. Indicating the axes with straws in my first piece was meant to draw attention to the viewing angles which expose the squares and triangles (Figure 5). However, based on how people engaged with the model at the gallery, it seems that looking "down" the axes is not instinctive.

Ideally, each of the seven sets of helices would have a distinct color that pops out when viewing the sculpture from a certain perspective. Alison Martin recently created an elaborate weaving of the gyroid surface, which does manage to show all seven helical sets with individual colors [6]. But doing so with a model that resembles the vertex-edge lattice of the TST, rather than the gyroid, may be more challenging — particularly with a modular system.

Another goal, for the sake of aesthetics, is to display a portion of the TST whose outer boundary is symmetric about all seven helical axes. Intuition suggests it possible to circumscribe a portion of the TST in a cuboctahedron or rhombic dodecahedron, two (mutually dual) polyhedra which share many properties with the TST. However, as the TST is chiral, such a boundary might only ever approximate true symmetry.

Finally, I aspire to display the infinitude of the full TST lattice. "Infinite" sculptures have been created previously by building a polyhedron with one-way mirrors for faces. But indeed, chirality precludes this approach — the helices would flip direction back and forth upon each mirroring. One compromise, namely on presentability, is to digitally render the infinite lattice with a technique such as raymarching, and explore it in virtual reality. If physically displaying the infinite TST is possible, the way to do so remains unclear.

Acknowledgements

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