A Papercrafted Fish Pattern on a Triply Periodic Polyhedron

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Abstract

We have used papercrafting to implement a pattern of fish on the regular triply periodic polyhedron {6,6|3}. We discuss some background for this pattern and a newly found relation to a previously designed patterned polyhedron.

Introduction

A main goal of our artwork is to place aesthetically pleasing tessellating patterns on hyperbolic surfaces such as triply periodic polyhedra. Figure 1 shows our latest patterned polyhedron — an Escher-inspired fish pattern on the regular infinite polyhedron {6,6|3} using papercrafting technology.

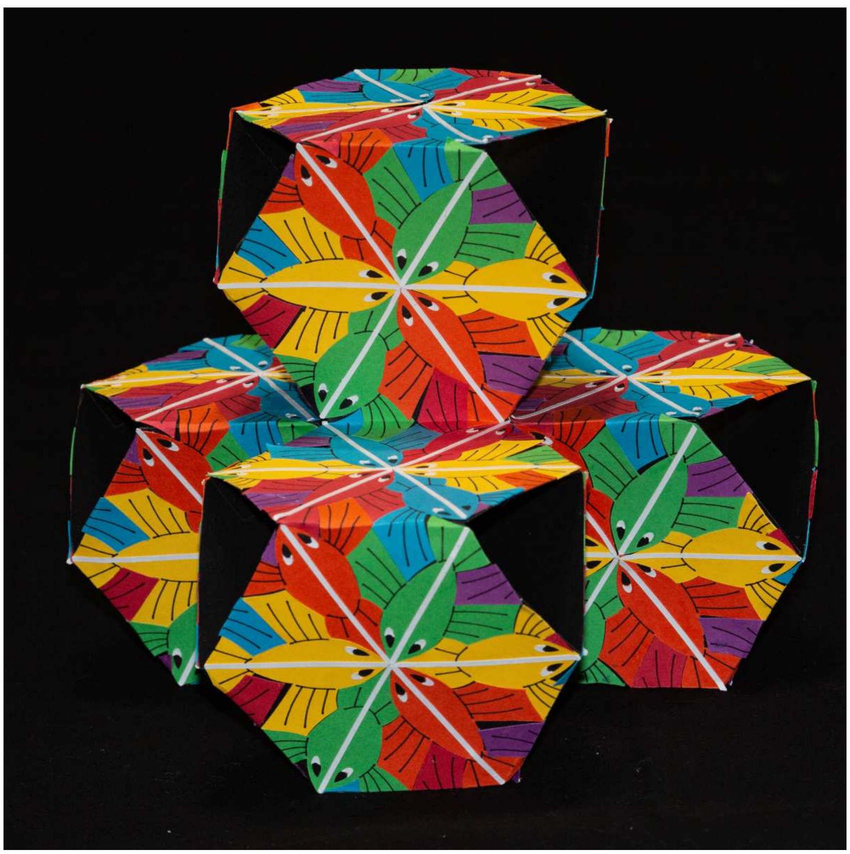

Figure 1: *Part of a papercrafted* {6,6|3} *polyhedron with an Escher-inspired fish pattern.*

Last year we designed a pattern similar to that of Figure 1 as a "proof of concept", and implemented it by hand [3]. The patterned polyhedron of Figure 1 is the "production" version, which was created by papercrafting technology, thus achieving a major goal in the Future Work section of [3].

In the next section, we review the history of M.C. Escher's two hyperbolic fish patterns *Circle Limit I* and *Circle Limit III*, and how he used the latter pattern to solve the aesthetic problems he saw in *Circle Limit I*. Then, after a brief review of triply periodic polyhedra, we discuss similar problems that we had early on in designing our own patterned polyhedra. Next we discuss a relationship we had not previously noticed between the latest fish-patterned polyhedron and a previous one. Finally we list conclusions and future work.

Escher's *Circle Limit I* and *Circle Limit III* Patterns

In 1958 Escher created his first "hyperbolic" pattern *Circle Limit I*, which can be considered to be a tessellation of the hyperbolic plane by angular black and white fish [5]. A bit more than a year later he created *Circle Limit III*, which solved aesthetic problems he saw in *Circle Limit I* [6]. Figures 2 and 3 show our renditions of those patterns, created by our own software [1]. Though *Circle Limit I* was Escher's first "circle limit" pattern, he was dissatisfied with it because it had these three shortcomings as he saw it:

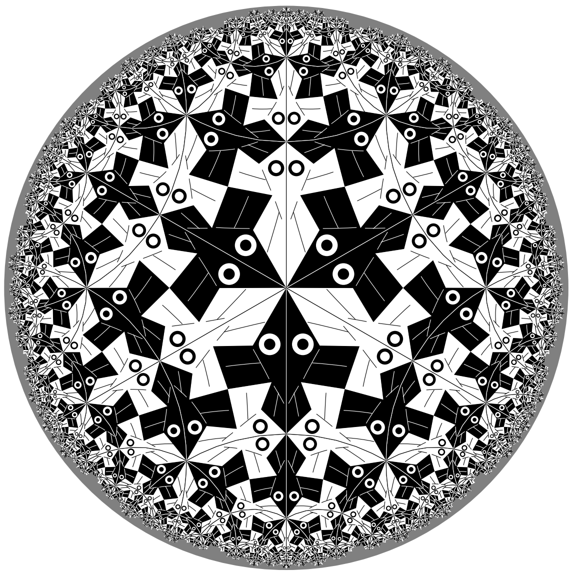

Figure 2: *Our rendition of* Circle Limit I.

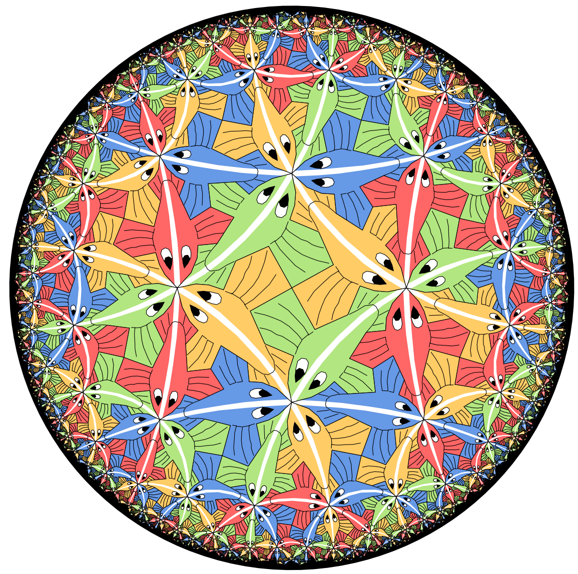

Figure 3: *Our rendition of* Circle Limit III.

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no "traffic flow" (Escher's words) in a single direction along the backbone lines.
3. The fish are very angular and not "fish-like" (we only know they are fish because Escher said they were).

In 1959 Escher solved these problems in his pleasing woodcut *Circle Limit III* [6]. Our rendition is shown in Figure 3. The fish are the same color along each backbone, they swim head to tail, and they are fish-shaped.

Infinite Polyhedra and Patterns on Them

Our main quest is to create repeating hyperbolic patterns and display them on geometric objects such as the Poincaré disk (as Escher did) or on negatively curved triply periodic polyhedra. The most regular such polyhedra are {4,6|4}, {6,4|4}, and {6,6|3}, where {p,q|r} is the Schläfli symbol denoting a polyhedron composed of p-gons (regular p-sided polygons) meeting q at each vertex, with regular r-sided polygonal holes.

About 10 years ago one of us started placing Escher-inspired patterns on such polyhedra [4]. Figures 4, and 5 show two of those patterns with Escher-inspired fish. The pattern of Figure 4 has all of the flaws that Escher saw in *Circle Limit I*; the pattern of Figure 5 has flaws 2 and 3. In 2021 we used papercrafting

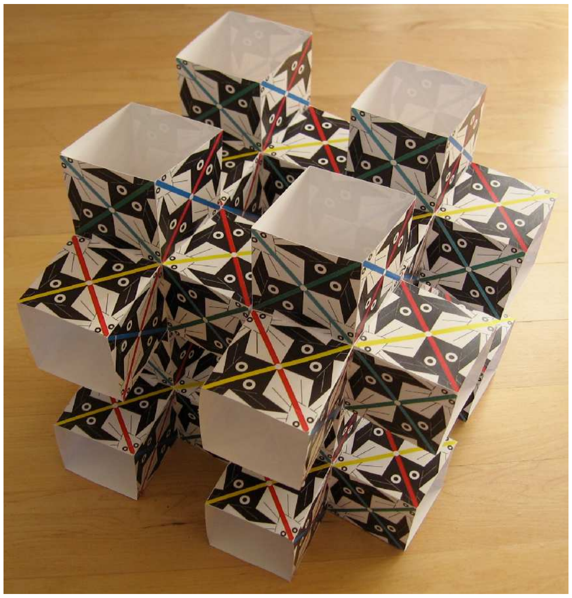

Figure 4: *Our 2012 fish pattern on the* {4,6|4} .

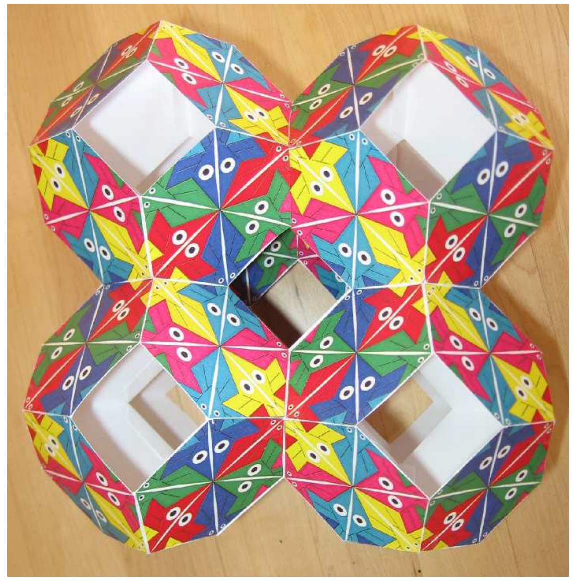

Figure 5: *A 2012 fish pattern on the* {6,4|4} .

technology to create a fish pattern on the {4,6|4} polyhedron, shown in Figure 6 below [2]. This pattern solved all of the problems we saw in our 2012 fish pattern except the "traffic flow" flaw. Last year we resolved that last problem by putting our hand-drawn fish on the {6,6|3} polyhedron [3].

We set the following restriction for our fish-patterned {p,q|r} polyhedra: the backbone lines must lie on embedded Euclidean lines that are not along p-gon edges. Thus we must use backbone lines that pass through the centers of the squares in the {4,6|4} and the centers of the hexagons in the {6,4|4} and {6,6|3} polyhedra. Of course Escher didn't have to contend with this restriction in his *Circle Limit* patterns.

This year we completed this ongoing project by making a papercrafted version of that pattern as shown in Figure 1. This was done using a Circuit Maker cutter/plotter to print on and cut the different pieces of colored paper. We used Floriani Craft 'N Cut software to create scalable vector graphics objects as input. The final polyhedron required 463 pieces of paper, which were glued in place by hand.

A Relationship Between Fish Backbone Lines on two Patterned Polyhedra

In the past, we have discussed the geometry of the backbone lines of fish on our patterned polyhedra {4,6|4} [2] and {6,6|3} [3]. As per Escher, we wanted to have all the fish along each backbone line be the same color. Moreover we also wanted all the fish on parallel lines in 3-space to be the same color, which was achieved. This produced six sets of parallel lines, one for each of the colors: red, orange, yellow, green, blue, and purple. The backbone lines on the {4,6|4} contain the diagonals of the square faces, so there are six sets corresponding to the pairs of diagonals on each of the three ways the faces are oriented in 3-space.

The {6,6|3} polyhedron is composed of truncated tetrahedra, all related to each other by translations in 3-space. As mentioned in [3], there are parallel backbone lines of fish of the same color passing through the centers of each pair of adjacent hexagons. Those fish are going opposite directions on the adjacent hexagons, as can be seen with the yellow fish in Figure 7: they are swimming left-to-right on the upper/horizontal hexagons and right-to-left on the downward slanted hexagons toward the front. Thus we can associate this color to the common tetrahedron edge of the adjacent hexagons.

Figure 6: *Our fish on the* {4,6|4} *(2021).*

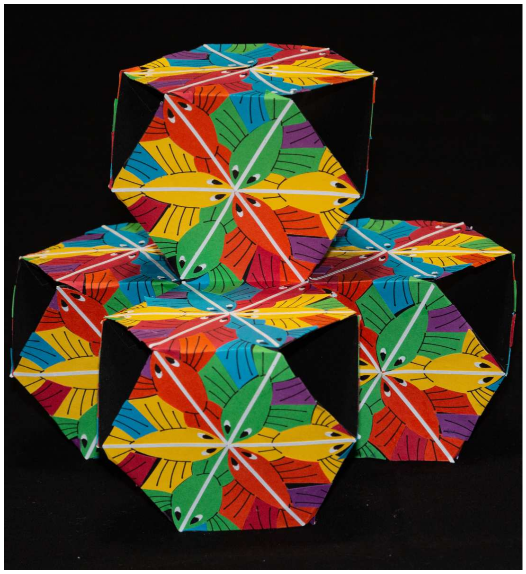

Figure 7: *Our fish on the* {6,6|3} *(2023).*

This leads to our new observation that the two patterned polyhedra {4,6|4} and {6,6|3} can be oriented in 3-space such that their sets of parallel backbones of each color are in the same direction. As can be seen by comparing Figures 6 and 7. This is because a regular tetrahedron (the modules of the {6,6|3} are truncated tetrahedra) can be inscribed in a cube with its edges coinciding with alternate diagonals of the cube.

Conclusions and Future Work

We have shown a papercrafted pattern on the {6,6|3} polyhedron in Figure 1 that fixes all of the problems of the polyhedron pattern in Figure 4, including Escher's "traffic flow" problem.

Another direction we would like to pursue is to render this pattern on Schwarz' D-surface, a triply periodic minimal surface that has the same topology as the {6,6|3} polyhedron.

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