# Curved-Crease Origami Spirals Constructed from Reflected Cones

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## Abstract

We describe two exact geometric constructions of *origami spirals* obtained by creasing a flat sheet of paper along $2n$ curves, alternating mountain and valley, where the 2D crease pattern and resulting 3D folding are $2n$-fold rotationally symmetric about the center. Both constructions use conical developable surfaces and planar creases. In one construction (conical spirals), the cone patches all share an apex (the center), effectively forming one big (creased) cone. In the second construction, inspired by David Huffman's "exploded vertex" designs, the cone apices are the vertices of a central regular polygon. Both constructions have planar creases and, in addition to their rotational symmetry, are reflectionally symmetric through the base plane.

## Introduction

Origami with curved creases forms beautifully intricate geometries using relatively few creases. One family of designs, which we call *spirals*, folds $2n$ rotationally symmetric curves, alternating mountain and valley [4][5]. While in many designs the curves meet at the center, David Huffman [2] introduced a variation with an "exploded" central vertex, replacing the central vertex by a regular $2n$-gon.

Origami spirals are related to origami flashers, a family of straight-crease rigidly foldable deployable structures that result from wrapping material around a polygonal base [3]. Their compact folded state makes them suitable for applications such as starshaders in space-exploration [1].

Material imperfections make it nontrivial to determine whether a real-world folded shape mathematically *exists*, i.e., preserves the intrinsic distances on the surface (no stretching or tearing). In this paper, we give two geometric constructions for spirals made from (generalized) cones with planar creases, resulting in shapes that are guaranteed to (mathematically) exist. We construct only the folded geometries, conjecturing that the constructed spirals do not have rigid-ruling folding motions.

## Developable Surfaces and Planar Creases

Developable surfaces are surfaces that can be obtained by bending a piece of flat material without stretching or tearing. They can be characterized as surfaces that contain a family of lines, the so-called *rule lines*, along which the surface's tangent planes are the same. There are three basic types of smooth developable surfaces — cones, cylinders, and tangent developables — and any developable surface is a combination of those. In applications, smooth developable surfaces are often approximated by their discrete counterparts [2][7]: developable planar quad and triangle strips, where one family of edges corresponds to the set of rulings, and the other to the discrete boundary curves. For brevity, we refer to both smooth developables and their discrete counterparts as (developable) *patches*.

Patches can be joined along their (smooth or discrete) boundary curves. In the special case that the combination is developable, that is, when the unrolled common boundaries are the same, this combination is called a *(curved) crease*, as it can be obtained by creasing the developed common boundary. Although the construction of such creases is in general challenging, the special family of *planar creases* is simpler to work with, making it well-suited for design. One way to construct planar creases is as follows: Given a patch $\Sigma$ that intersects a plane $\Pi$, use $\Pi$ (specifically, one connected component of $\Sigma \cap \Pi$) to split $\Sigma$ into two subpatches, and reflect one of the subpatches through $\Pi$. Reflection preserves connectivity of the surface and its development, and (typically) results in a crease that is contained in the reflecting plane $\Pi$.

## Triangle Wreath

The first step in our construction is to position $2n$ congruent triangles in a rotationally symmetric way; see Figure 1. Precisely, a *triangle wreath* consists of $2n$ congruent triangles $(\triangle_0, \triangle_1, \ldots, \triangle_{2n-1})$ such that cyclically consecutive triangles $\triangle_i, \triangle_{(i+1) \bmod 2n}$ share a vertex $\mathbf{P}_i$ and are related by the rotation $\mathbf{R}_n$ by $\frac{\pi}{n}$ about the $z$-axis and the reflection $\mathbf{M}_{xy}$ through the $xy$-plane:

$$\triangle_{(i+1) \bmod 2n} = \mathbf{M}_{xy}\big(\mathbf{R}_n(\triangle_i)\big) \quad \text{for all } 0 \le i < 2n.$$

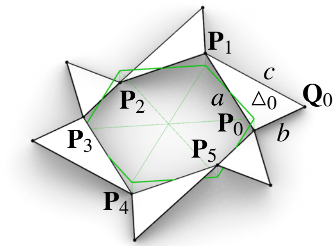


**Figure 1:** *Triangle wreath.*

**Lemma 1.** *Given a triangle $\triangle = (a, b, c)$, up to rotation about the z-axis, there exists a two-parameter family of triangle wreaths in which the endpoints of triangle edge a are joined.*

*Proof.* Let $\mathcal{P} = (\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{2n-1})$ be the polyline formed by consecutive vertices that coincide with two triangles in a triangle wreath. Because $\mathbf{P}_0$ and $\mathbf{P}_1$ are related by rotation $\mathbf{R}_n$ and reflection $\mathbf{M}_{xy}$, their coordinates can be written as $\mathbf{P}_0 = \big(r\cos\frac{\pi}{n}, -r\sin\frac{\pi}{n}, -h\big)$ and $\mathbf{P}_1 = \mathbf{M}_{xy}\big(\mathbf{R}_n(\mathbf{P}_0)\big) = \big(r\cos\frac{\pi}{n}, r\sin\frac{\pi}{n}, h\big)$ for some $h > 0$ and $r > 0$. Because $|\mathbf{P}_0 - \mathbf{P}_1| = a$, it follows that $r = \frac{1}{2}\sqrt{a^2 - 4h^2} \Big/ \sin\frac{\pi}{n}$ for $h \in [0, \frac{a}{2})$. Thus we have one degree of freedom for the configuration of the polyline $\mathcal{P}$. In addition, we have another degree of freedom corresponding to the location of the third vertex of the first triangle $\triangle_0 = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_0)$ of the wreath, corresponding to a rotation about $\mathbf{P}_0\mathbf{P}_1$. □

## Curved Spiral Construction

Now we describe how to construct a conical or an exploded-vertex spiral by attaching cones to the triangles of a triangle wreath and reflecting appropriately. In the following, let $\Gamma$ be a (smooth or discrete) cone with developed opening angle $\frac{\pi}{n}$, specified by an apex $\mathbf{V}$ and a (smooth or discrete) boundary curve $\mathbf{C}(t)$ where $t \in [0, 1]$. Without loss of generality, we assume that the first and last ruling of $\Gamma$ are of the same length, $1 = |\mathbf{V} - \mathbf{C}(0)| = |\mathbf{V} - \mathbf{C}(1)|$. The two proposed constructions of spirals with $2n$ cones consist of three steps:

- *Step 1:* Use points on $\Gamma$ to define a triangle $\triangle = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_0)$ as follows. Then use Lemma 1 and two input parameters to construct a wreath of triangles congruent to $\triangle$. Position a copy $\Gamma_0$ of $\Gamma$ at the corresponding points on the first triangle $\triangle_0$ of the triangle wreath. Define a second cone $\Gamma_1 = \mathbf{M}_{xy}\big(\mathbf{R}_n(\Gamma_0)\big)$.
  - *Conical spiral:* Let $\mathbf{P}_0 = \mathbf{C}(0)$; $\mathbf{P}_1 = \mathbf{C}(1)$; and $\mathbf{Q}_0 = \mathbf{V}$.
  - *Exploded-vertex spiral:* Let $\mathbf{P}_0$ be some non-apex point on the first cone ruling, that is, $\mathbf{P}_0 = (1-t)\mathbf{V} + t\mathbf{C}(0)$ for $t \in (0, 1)$;[1] $\mathbf{P}_1 = \mathbf{V}$; and $\mathbf{Q}_0 = \mathbf{C}(1)$.
- *Step 2:* Fold $\Gamma_0$ along a planar crease to connect a subpatch of $\Gamma_0$ with the other subpatch of $\Gamma_1$. Fold the other cone $\Gamma_1$ analogously.
  - *Conical spiral:* Let $\Pi_0$ be the bisecting plane of $\mathbf{P}_0$ and $\mathbf{P}_1$. Split $\Gamma_0$ along $\Pi_0$ into two subpatches, and reflect the subpatch containing the apex through $\Pi_0$.

[1]Because we extend the cones in Step 3 anyhow, the choice of $t$ influences the scale. In our implementation, we use $t = \frac{1}{4}$.

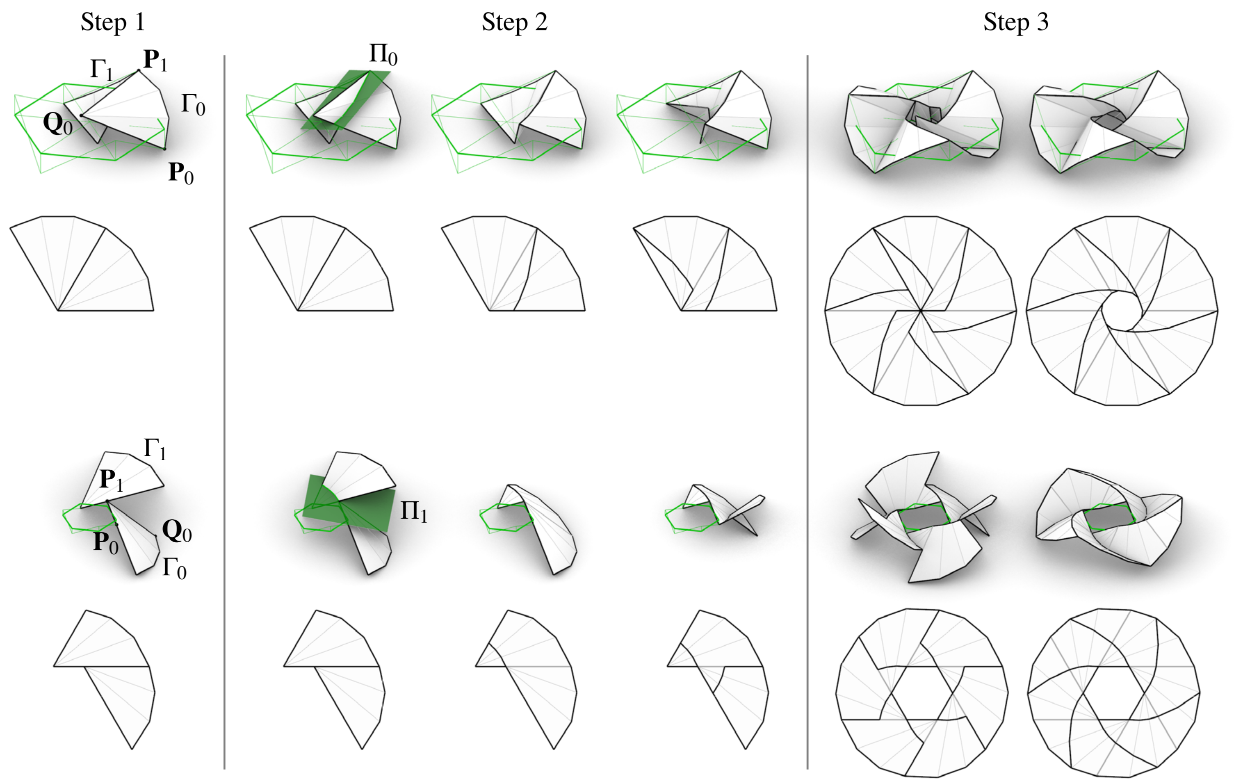


**Figure 2:** *Construction steps for the conical spiral (top) and the exploded-vertex spiral (bottom).*

  - *Exploded-vertex spiral:* Define $\Pi_1$ to be the bisecting plane of $\mathbf{C}_0(1)$ and $\mathbf{C}_1(0)$. Split $\Gamma_1$ along $\Pi_1$ into two subpatches, and reflect the subpatch not containing the apex by $\Pi_1$.

  Both reflections join a subpatch of $\Gamma_1$ with the other subpatch of $\Gamma_0$, resulting in a connected surface.

- *Step 3:* Arrange copies of the two cones in a polar array to obtain a closed ring of $2n$ creased cones. If possible, extend the creases to neighboring cones by using their incident planes for splitting and reflecting, and trim the surfaces appropriately.

Note that this construction might fail for unrealistic parameter combinations, or result in intersecting cones. Upon success, this construction results in closed developable rings of patches joined along creases. Even if the involved cones are smooth, the connection between $\Gamma_0$ and $\Gamma_1$ might not be tangent continuous. In many cases, tangent continuity can be achieved by tweaking the parameters of the wreath construction. If the polyline of the wreath is planar, we can close the center of the spiral with a planar polygonal face.

We conjecture that the conical spiral construction can be generalized to patches other than cones, in which singularities of the surface might need special treatment.

## Software Implementation

We implemented the described construction as an interactive design tool for Grasshopper / Rhino. To construct smooth or discrete cones with appropriate opening angle, we use the sliding developable method proposed in [6] and its implementation described in [7]. The position of the first triangle $\triangle_0 = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_0)$ of the wreath can be influenced by the user with two angles: angle $\varphi$ describes the inclination of $\mathbf{P}_0\mathbf{P}_1$ with respect to the $xy$-plane, while angle $\psi$ describes the angle between $\triangle_0$ and the plane spanned by the $x$-axis and $\mathbf{P}_0\mathbf{P}_1$. We use an external plug-in named Goat [8] to optimize for tangent continuity.

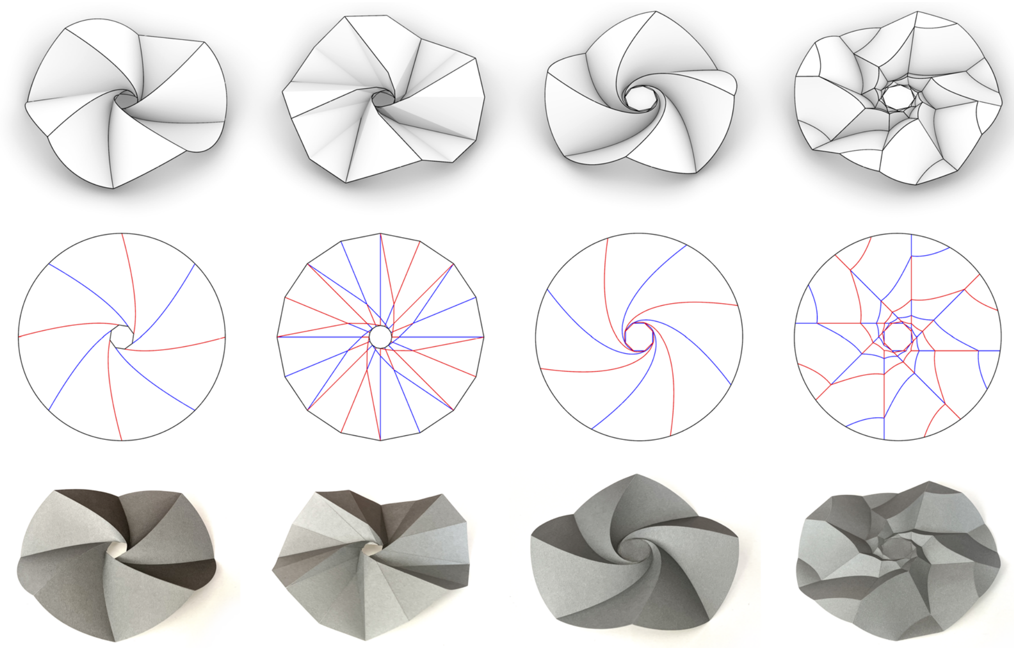

**Figure 3:** *Example designs using our constructions of two conical spirals (left) and two exploded vertex spirals with a closed polygonal center (right). Top row: Designed geometry. Middle row: Constructed crease pattern. Bottom row: Paper-folded shapes.*

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