How to Build an Origami Nine-Color Map on a Genus-Three Torus

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Abstract

This paper gives directions for building a 9-color map on a genus-three torus using Sonobe units. The resulting origami sculpture is a three-dimensional proof that the Heawood bound for maps on a genus-three surface can be realized.

Introduction

A *map* on a surface is a division of the surface into regions, each of which is topologically a disk. In 1890, Percy Heawood [3] showed that *n* colors, where *n* is the largest integer less than or equal to ½ $(7 + \sqrt{1 + 48g})$, are sufficient to properly color any map on an orientable surface of genus *g* > 0. In particular, a map on a three-holed torus (*g* = 3) can require at most nine colors. But Heawood did not show that there was a map that requires *n* colors for surfaces on a genus-*g* surface except for genus-one. In 1891, Lothar Heffter [4] was the first to publish a map on a genus-three surface that required nine colors. Heffter used tables of numbers to describe this map. Over the years many people have struggled to design physical models of maps on higher genus surfaces that would give more insight into the structure of these maps [1][2]. In 2021, Carlo Séquin designed several maps on a genus-three surface that required nine colors [5]. In 2022, I designed and built a three-dimensional origami sculpture (Figure 1) based on Séquin's map shown in Figure 2.
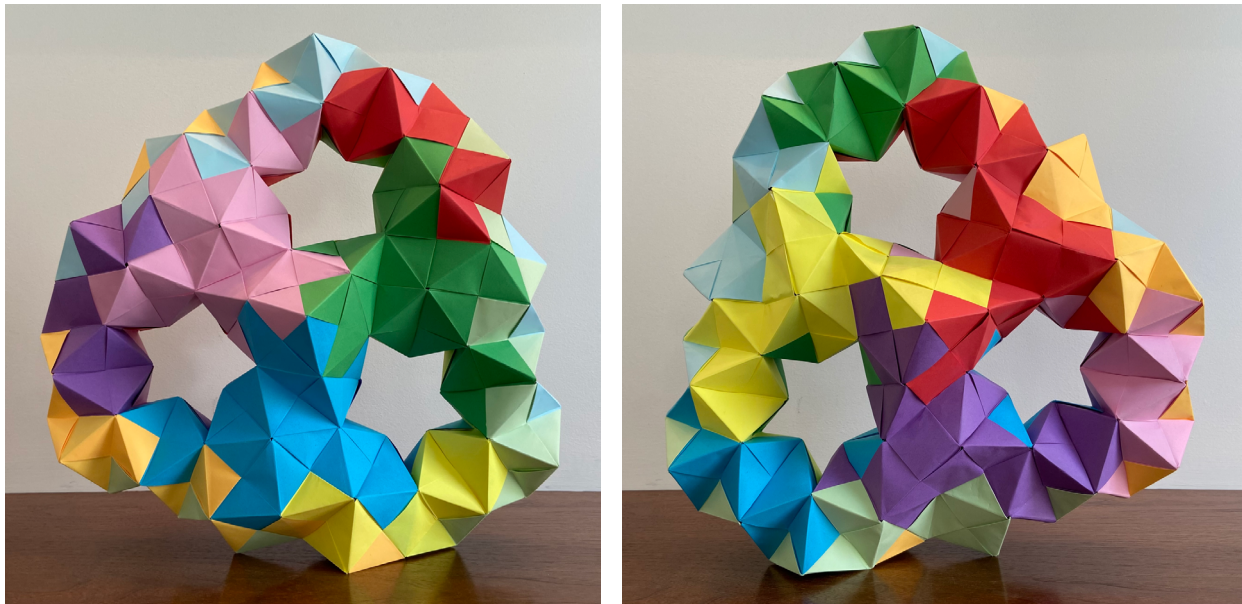

Figure 1: *Sonobe unit model of a nine-color map on a three-holed torus, front (left) and back (right).*
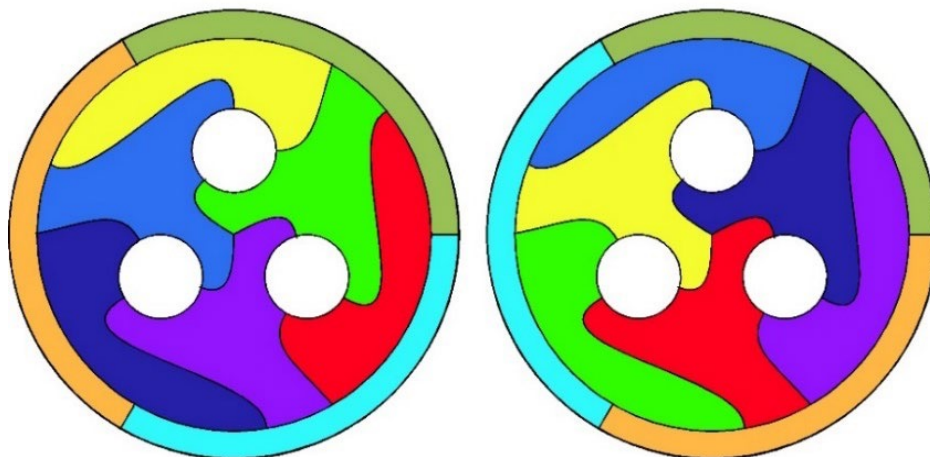
Figure 2: *Séquin's model of a nine-color map on a three-holed torus, front (left) and back (right). Some of the shared boundaries between different color regions occur on the outside edge or through a hole.*

Instructions and Diagrams

My 2022 Bridges workshop paper, *Modular Origami Map Coloring Models* [6], gives instructions for building a model of a seven-color map on a torus and an eight-color map on a double torus. This paper picks up where that paper left off. Beginning directions for understanding and building this model can be found in that paper.

This model is made from 129 Sonobe units with eleven variations of additional folds. The unit types are labeled by letter on the diagrams. Folds for unit types A–E can be found in [6], while the folds for the remaining six types of units are shown in Figure 3. Table 1 gives the number of units of each type by color. Label units on both flaps as in Figure 3.

Units are always connected by inserting a unit into the unit in the diagram to its immediate right in the clockwise direction. Following the diagram in Figure 2, build the front of the model first, starting at the center triangle made of J-units. Next work your way to the outer edge of the front diagram. Turn the structure over and add the units on the back around the holes. Then connect the holes to the center triangle on the back made of L-units. Finally, complete the structure by joining the units around the outer edge of the front and back.

Table 1: *Number of units by type and color.*

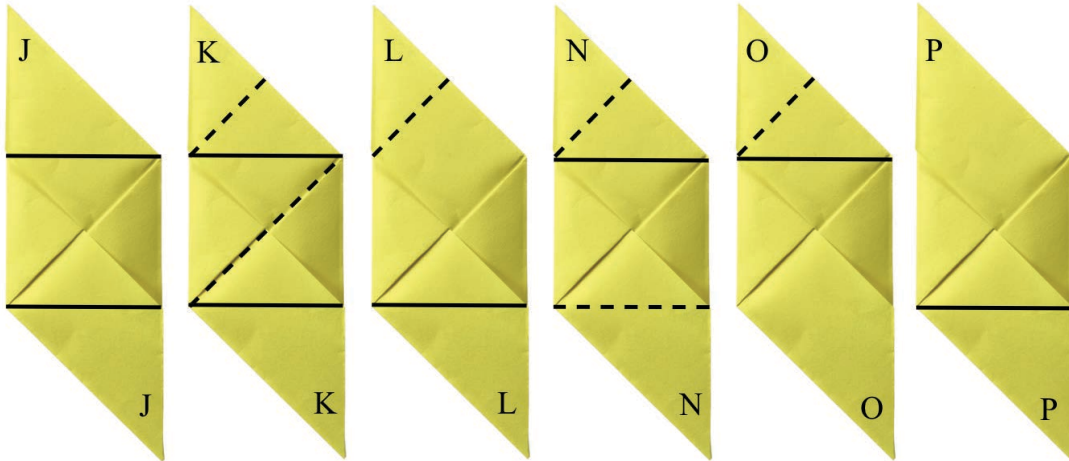| Unit Type:	A	B	C	D	E	J	K	L	N	O	P	Total
Dark Blue	4	6	2		1	2	1					16
Dark Green	4	6	2		1	2	1					16
Pink	4	6	2		1	2	1					16
Red	2	4					4	3		1		14
Yellow	2	4					4	3		1		14
Purple	2	4					4	3		1		14
Light Blue	2	3	1	1		1	1	1	1	1	1	13
Light Green	2	3	1	1		1	1	1	1	1	1	13
Orange	2	3	1	1		1	1	1	1	1	1	13

Figure 3: *Additional Sonobe unit variations for the nine-color torus. Mountain folds are solid lines and valley folds are dashed lines.*
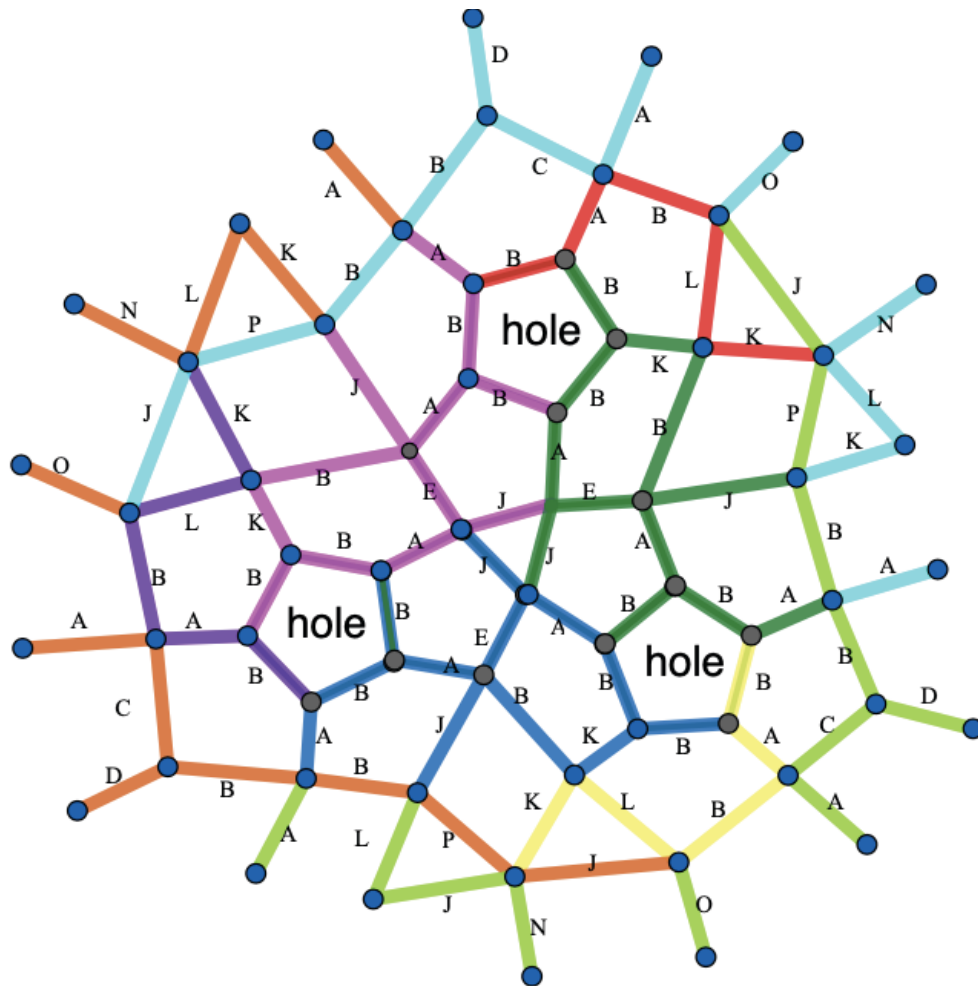

Figure 4: *Diagram for constructing the front of an origami model of a nine-color map on a three-holed torus.*
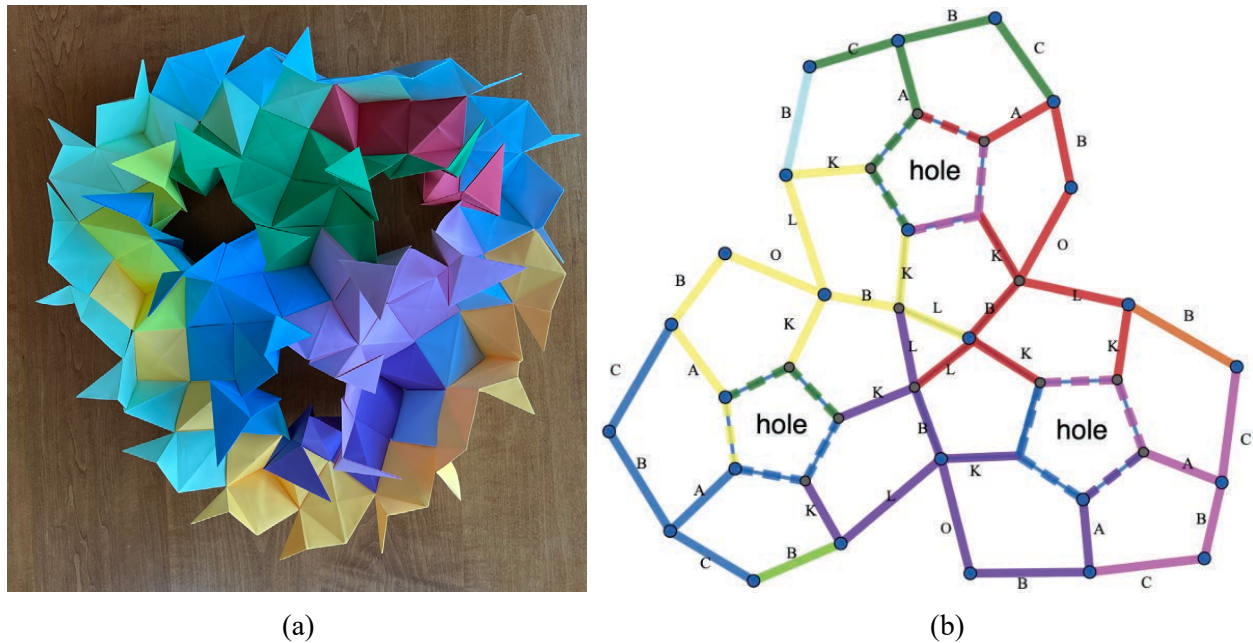
(a)   (b)

Figure 5: *(a) The model after constructing the front and turning it over but before adding the back units.*
(b) Diagram for constructing the back of the model. Dashed edges are units from the front of the model and are only included on this diagram for reference.

Summary and Conclusions

Designing this model took several months and was quite challenging. After learning how Sonobe units could be used to make a torus, I drew the graph for that origami model and used it to design a seven-color map on that structure [6]. I then figured out how to join two Sonobe tori to make a genus-2 surface with an 8-color map. Finally, I challenged myself to make a 9-color genus-3 origami map. The process for designing these structures is similar to sculpting with other materials. But Sonobe units are a crude medium that can be difficult to shape. By trial-and-error, I was able to find an arrangement of units that formed both the correct surface and a complete 9-color map. I called my finished sculpture "Cloud Nine" for the satisfaction I felt on being able to successfully make this piece.

References

[1] E. Baker and E. Torrence. "Fabric Models of Maximal Complete Maps on a Three-holed Torus." Bridges Conference Proceedings, Halifax, Nova Scotia, Canada, July 27–31, 2023.

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[3] P. J. Heawood. "Map Colour Theorem." Quarterly Journal of Pure and Applied Mathematics, No 24, pp. 332–338, 1890.

[4] L. Heffter. "Ueber das Problem der Nachbargebiete." Mathematische Annalen. Vol 38, pp. 477–508, 1891.

[5] C. Séquin. "Easy-to-Understand Visualization Models of Complete Maps." Bridges Conference Proceedings, Halifax, Nova Scotia, Canada, July 27, 2023.

[6] E. Torrence. "Modular Origami Map Coloring Models." Bridges Conference Proceedings, Helsinki, Finland, August 1– 5, 2022, pp. 507–514.
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