Fabric Models of Maximal Complete Maps on a Three-Holed Torus

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Abstract

In 1890, Lothar Heffter described a complete map with nine regions on a surface of genus three. Heffter used a table of numbers to present his map, which makes it very hard to visualize. This paper shows how we used Heffter's numerical description to sew a physical model of Heffter's map that clearly shows it is indeed drawn on a surface of genus three. The sewn model also reveals other interesting features of this map. In addition, we discuss models we have sewn of two other distinct nine region complete maps on a genus three surface.

Introduction

A *map* on a surface is a division of the surface into regions, each of which is topologically a disk. A map is *complete* if every pair of regions shares a boundary line. A map is *properly colored* if regions that share a boundary line are different colors. Note that a complete map with n regions requires n colors to be properly colored. In 1890, Percy Heawood [2] showed that n colors, where n is the largest integer less than or equal to $\frac{1}{2}(7 + \sqrt{1 + 48g})$, are sufficient to properly color any map on an orientable surface of genus $g > 0$. For example, all maps on a torus ($g = 1$) can be properly colored with at most seven colors. A map on a two-holed torus ($g = 2$) can require at most eight, and a map on a three-holed torus ($g = 3$) can require at most nine. A map on a genus-g surface is *maximally complete* if it has n regions. Heawood included a drawing of a complete 7-region map on a torus in his paper. But he simply stated that “for highly connected surfaces it will be observed that there are generally contacts enough and to spare for the above number of divisions each to touch each” [2]. In his 1891 paper [3], Lothar Heffter pointed out that while Heawood had shown the maximum number of colors needed to properly color such maps, this hand waving was not enough to show that a maximally complete map exists for every genus-g surface for $g > 1$. Heffter’s paper included maximally complete maps for surfaces of genus 0 through 6. But he used tables to describe these maps (Figure 1(a)). While the use of tables was a brilliant innovation in the study of maps on higher genus surfaces, it is frustratingly opaque for understanding the structure of these maps.

$n = 9, \quad p_9 = 3, \quad \alpha_9 = 3, \quad F = 23.$

1)	2 3 4 5 6 7 8 9
2)	6 1 9 7 3 5 8 4
3)	4 1 2 7 6 8 5 9
4)	5 1 3 9 6 2 8 7
5)	6 1 4 7 9 3 8 2
6)	7 1 5 2 4 9 8 3
7)	8 1 6 3 2 9 5 4
8)	9 1 7 4 2 5 3 6
9)	2 1 8 6 4 3 5 7

22 Dreiecke und das Sechseck [3 1 2 6 5 2].

(a)

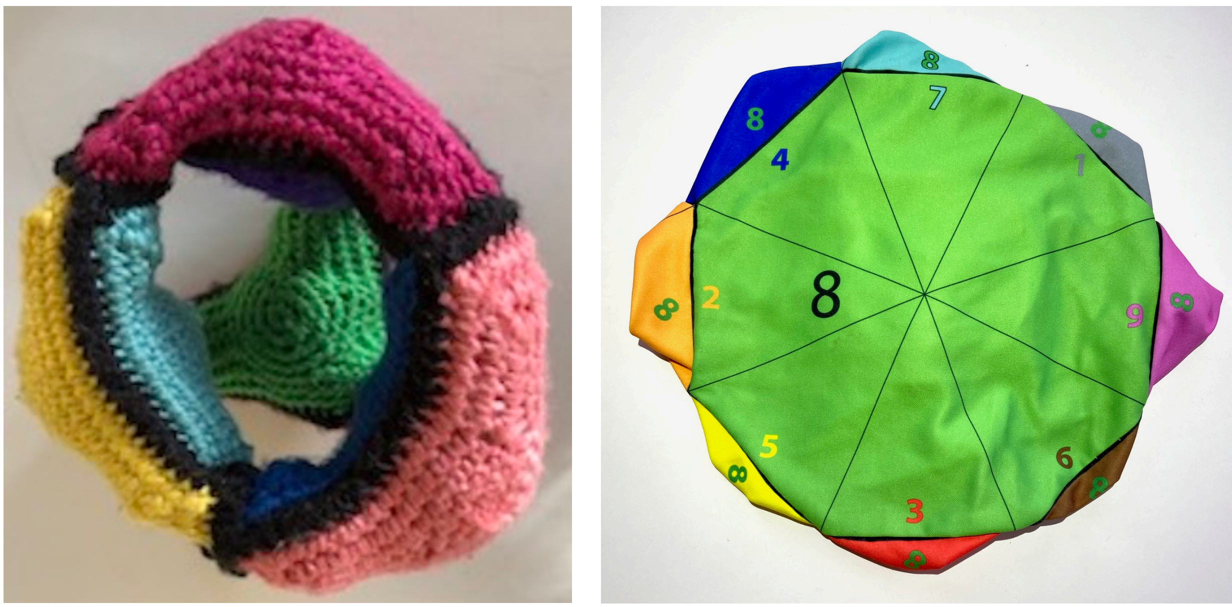
(b)

Figure 1: *(a) Heffter’s table for a nine region complete map on a genus-3 surface from his 1890 paper. (b) Our fabric model of Heffter’s nine region map showing the 3 holes in the surface.*

Making Fabric Models: Insight and Constraints

We were motivated to make a fabric model of Heffter's 9-color map after reading Moira Chas' AMS feature column "Crochet Topology" [1]. Chas included a crocheted model of a complete, but not maximal, eight-region map on a genus-3 surface (Figure 2(a)). She then gave Heffter's table and wrote, "We encourage the reader to try to construct a more geometrically intelligible complete map with nine regions on a genus three surface, and perhaps to render it in crochet."

Figure 1(b) shows our fabric model of Heffter's 9-color map arranged to show the surface has three holes. Figure 2(b) shows the same model arranged to show that the green region shares a boundary line with the other eight regions. The model can be manipulated into a similar layout for each of the nine regions.



(a) (b)

Figure 2: *(a) Moira Chas' crocheted eight color map on a genus 3 surface. (b) Our 9-color fabric model arranged to show how the eight other regions share a boundary line with the green region.*

To make our fabric model of Heffter's map we designed a flat template (Figure 3) using Heffter's table. Hoping to make it more "geometrically intelligible," we wanted a model in which every region had the same shape and every boundary was the same length. Hence, we made every region a regular octagon. Following Heffter's table we labeled the edges of the nine octagonal regions with the number and color of the other region to which each edge is joined. For example, the green region is color 8. Beginning at the edge near the pink region in Figure 2(b), the edges of region 8 are numbered 9, 1, 7, 4, 2, 5, 3, 6 in the counterclockwise direction following line 8 of Heffter's table. We had the flat template printed on Performance Piqué fabric by Spoonflower [5]. We then cut out the regions and sewed them together following the labels.

Our initial attempts to sew the regions together was complicated by a strange feature of Heffter's map. The text below the table in Figure 1(a) translates to "22 triangles and the hexagon [3 1 2 6 5 2]." This map has 22 vertices with valence-3 and one vertex with valence-6. But the vertex with valence-6 has a repeating region, region 2 (see Figure 4(a)). This creates an unusual and surprising feature of the map that makes this model confusing and difficult to sew. Region 2 needs to be folded over and its opposite vertices connected to form this valence-6, five region vertex. By trial and frustration we discovered it is best to sew this strange vertex first. But after that vertex is connected, the rest of the map is challenging to sew because the resulting surface is particularly twisted. We had hoped to make a model that could be stuffed and would be an obvious 3-holed torus. Stuffing only exacerbated the twisting. We found that an unstuffed model is best for studying the features of the map.

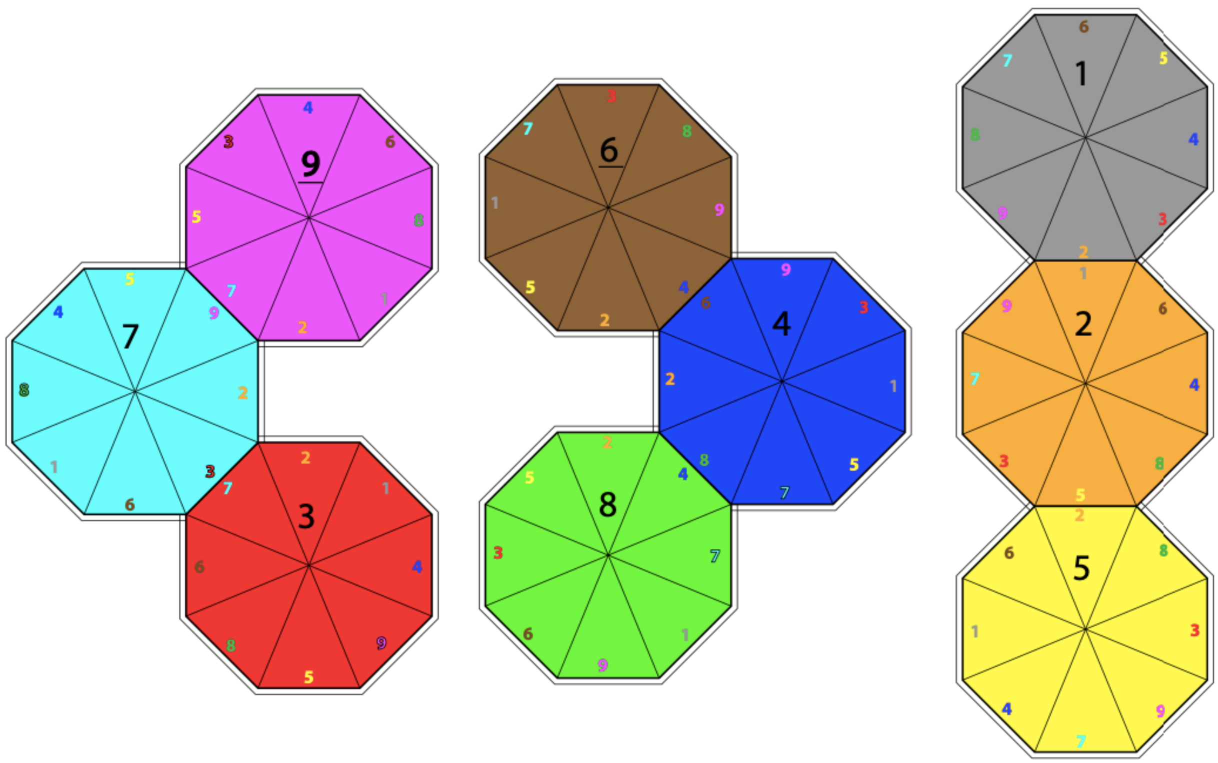


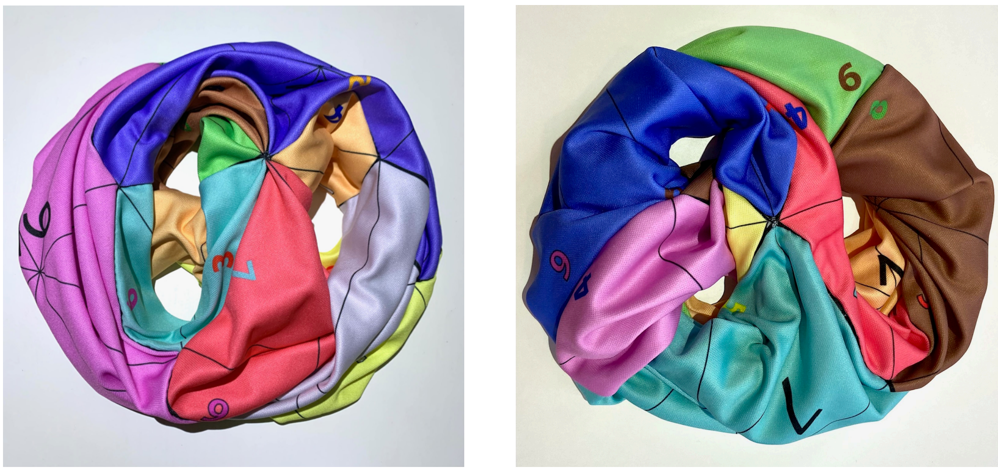
Figure 3: *Template for printed fabric used to sew a model of Heffter's 9-color map.*

(a) (b) (c)

Figure 4: *(a) Detail of the valence-6 vertex in Heffter's map where five regions meet.*
(b) Detail of the valence-6 vertex in Séquin's map where six regions meet.
(c) One of the three valence-4 vertices in Séquin's second map.

Models of Maps by Carlo Séquin

In 2021, Carlo Séquin designed two maximal complete graphs representing 9-color genus-3 maps that are distinct from Heffter's [4]. Like Heffter's map, Séquin's first map has one valence-6 vertex. But Séquin's valence-6 vertex joins six different regions instead of five (Figure 4(b)). Séquin's second 9-color map has three valence-4 vertices (Figure 4(c)). All other vertices for both these maps have valence three. We used Séquin's graphs to design flat templates for these maps as well, had them printed by Spoonflower, and sewed them into genus-3 surfaces (Figure 5). Our templates for Heffter's map as well Séquin's two 9-color maps are available in the Supplement to this paper and on Spoonflower [5].



(a) (b)

Figure 5: *(a) Séquin's map with one valence-6 vertex. (b) Séquin's map with three valence-4 vertices.*

Summary

These sewn models are elegant visual proofs of the correctness of the maps. There is no substitute for a model you can hold in your hands and manipulate to verify its properties. We invite you to print our map templates on fabric and sew your own models to explore.

Acknowledgements

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References

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[4] C. Séquin. "Easy-to-Understand Visualization Models of Complete Maps." Bridges Conference Proceedings, Halifax, Nova Scotia, Canada, July 27–31, 2023.

[5] Spoonflower www.spoonflower.com – search for elliebaker and order a full yard for each map.