Oh What a Complex Rug We Weave When First We Color Then Perceive

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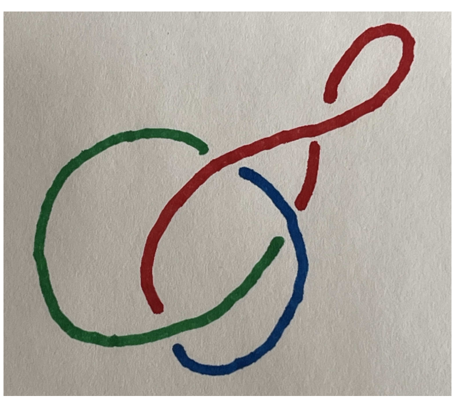
Abstract

This paper reports on some surprising patterns that emerge by obeying a simple mathematical rule, borrowed from knot theory, for tricoloring a square weave of "ribbon" or "thread," starting from specified "fringe" conditions across the top and left edges. We have no proofs as yet; indeed, it's a challenge just to describe what we see in full mathematical detail. But clear hints of Sierpinski arise . . .

A Knotty Question Looms

A useful technique in knot theory is to tricolor the connected arcs ("strands") that constitute the two-dimensional projection of a knot. The coloring rule requires that either one or three colors meet at each crossing: the two "underpassing" strands either match the overpassing strand's color or all three strands have different colors, as indicated in Figure 1 for a slightly tweaked trefoil knot. Among other nice properties, tricolorability is an indicator that a knot cannot be "unknotted" [1, pp. 22–27].

It occurred to one of the authors (BC) to see what would happen if the same rule were applied to the segments of a "basket weave," starting from a prespecified set of colors for the top and left fringes of the vertical "warp" and horizontal "weft," imagined to continue infinitely down and to the right.



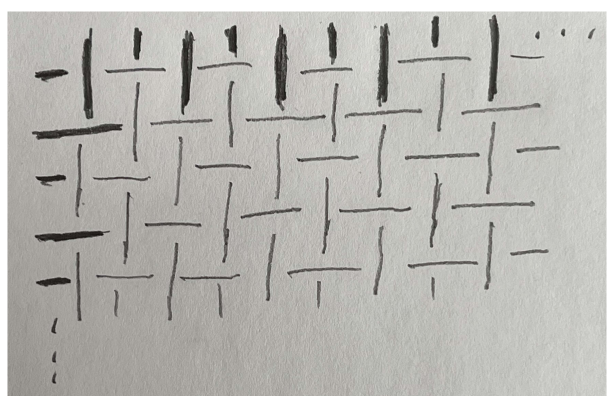


Figure 1: *A tricolored trefoil knot (left). In a basket weave (right), once the fringe strands (thicker lines) are assigned colors, the tricoloring rule determines all other colors.*

He began by simply cycling Red, Blue, and Yellow along each fringe. He then started carefully coloring the adjacent threads one by one, aware that any errors would propagate throughout the weave. He expected the result to be similarly simple, cyclic, and therefore dull, both mathematically and aesthetically.

He was wrong.

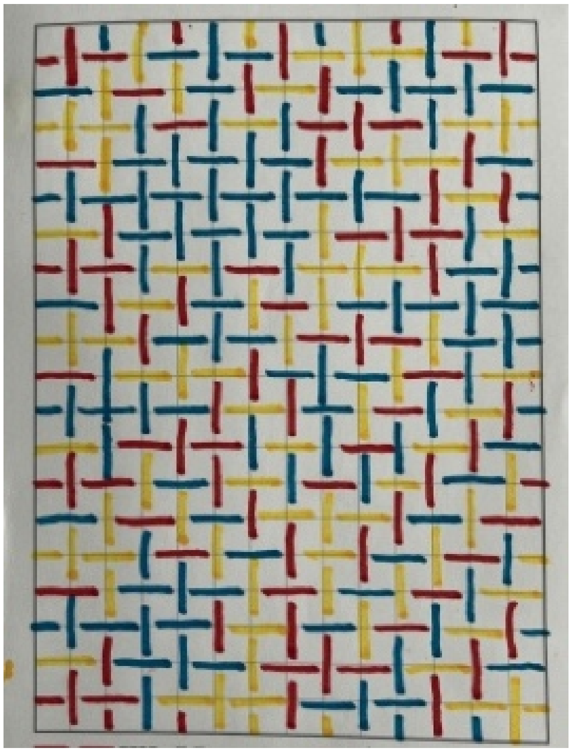


Figure 2: *Hand-crafted carpet with cyclic fringe*

BC's hand-drawn experiment (Figure 2) on a gridded scratchpad, with space for 19 wefts and 13 warps, looked anything but simple or cyclic. The result looked more random than patterned, including an unexpected blob of solid blue occurring fairly near the upper left hand corner. But clearly nothing truly random is going on here, since the prespecification of the fringes is simply cyclic and the rule for propagating colors is rigidly deterministic. Equally obviously, a 13×19 "carpet" is still fairly small; perhaps a larger grid would reveal some underlying simple, cyclic pattern.

Carpet Cleaning

Enter the second author (PZ), who enlisted *Mathematica* to do some industrial-strength weaving. PZ made two notable changes. One is minor: He changed Yellow to Green—which happens to accord with knot theorists' convention for tricoloring. The other is more substantial: Mainly for *Mathematica*l convenience, he fattened BC's threads into "ribbons," to produce a basket weave, with square overcrossings. While we remain agnostic as to the "best" choice of colors (we defer to Albers [2] and other experts on such matters), the threads-to-ribbons transformation turns out to have certain advantages.

The *Mathematica*-zation of a tricolored basket weave with prespecified fringes along the top and left goes as follows. First we associate the colors Red, Blue, Green with the numbers 0, 1, and 2 mod 3. Then, for any particular n, say $n = 27$, we create an $(n + 2) \times (n + 2)$ "checkerboard" of small squares, with rows numbered -1 to n, reading down, and columns -1 to n reading from left to right. Little squares get coordinates in the obvious way: $(11, 23)$ is in the row numbered 11 and the column numbered 23. Next, we color (i.e., assign numbers $c_{i,j}$ mod 3 to) the first two squares of the leftmost and topmost rows and columns. This represents our prespecified "fringe" conditions in a way that makes it easy to reformulate the tricoloring rule for the rest of the carpet ($i, j \geq 1$), which boils down to this *Mathematica*-friendly formula:

$$c_{i,j} = \begin{cases} -c_{i-1,j} - c_{i-2,j} \mod 3 & \text{if } i + j \text{ is even} \\ -c_{i,j-1} - c_{i,j-2} \mod 3 & \text{if } i + j \text{ is odd} \end{cases}$$

This formula encodes our desired coloring rule in that each square's color is determined by two immediate neighbors: "from above" for "even" squares (the black ones on a checkerboard, say, which correspond to "warp" ribbons) and "from the left" for "odd" (or "weft") squares. Note that $c_{i,j}$ is chosen so that the three numbers sum to 0 mod 3. This agrees with our tricoloring rule: three numbers sum to 0 mod 3 if and only if they are all the same mod 3 or are all different mod 3.

Perhaps unsurprisingly (but nonetheless to the authors' surprise), powers of 3 seem to play a key role in the patterns that emerge. Here's what *Mathematica* wove from BC's cyclic fringe conditions for $n \times n$ square carpets with $n = 27$, 81, 243, and 729:

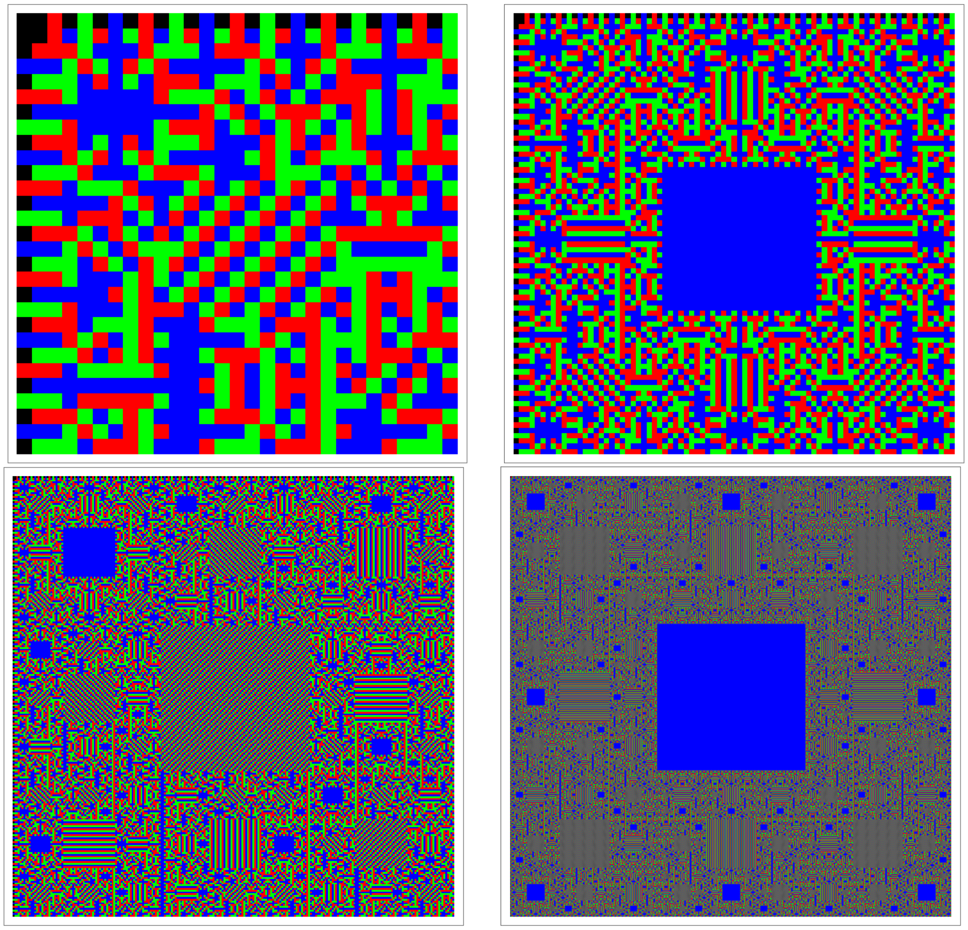


Figure 3: *Cyclic fringe conditions at scales* 27×27*,* 81×81*,* 243×243*, and* 729×729*. (Black portions of the fringe indicate squares that play no role in tricoloring the interior.)*

The patterns in Figure 3 suggest that the cyclic Red/Blue/Green fringes produce a Sierpinski-esque carpet. Most prominently, the small blob of solid blue seen in Figure 2 establishes itself at larger scales as a distinctive square of some regular pattern occupying (roughly) the middle ninth of each $3^n \times 3^n$ square carpet, with similar regularly patterned squares occupying the middle ninths of the eight blocks surrounding it. Some of the distinctive squares are solid blue; others are multi-colored stripes in horizontal, vertical, and diagonal directions. In particular, a solid blue middle ninth seems to occur for even powers of 3. None of this is rigorously certain, but we would be surprised if the observed patterns don't persist.

Loose Ends

One thing we can say with certainty is that if all fringes have only one color, say Green, then the entire carpet will be Green. Indeed, this modest observation prompted an experiment that offers a tantalizing glimpse at underlying structure: Starting from all-Green fringes, we make two tiny tweaks (or, arguably, mistakes), recoloring the first warp and the first weft fringe elements Red. Which carpet squares, we wondered, will change color from their original Green? With Purple (more or less the sum of Blue and Red) indicating squares that change color, and White indicating squares that don't, we get the striking Sierpinski-esque "difference carpet" shown at two scales in Figure 4.

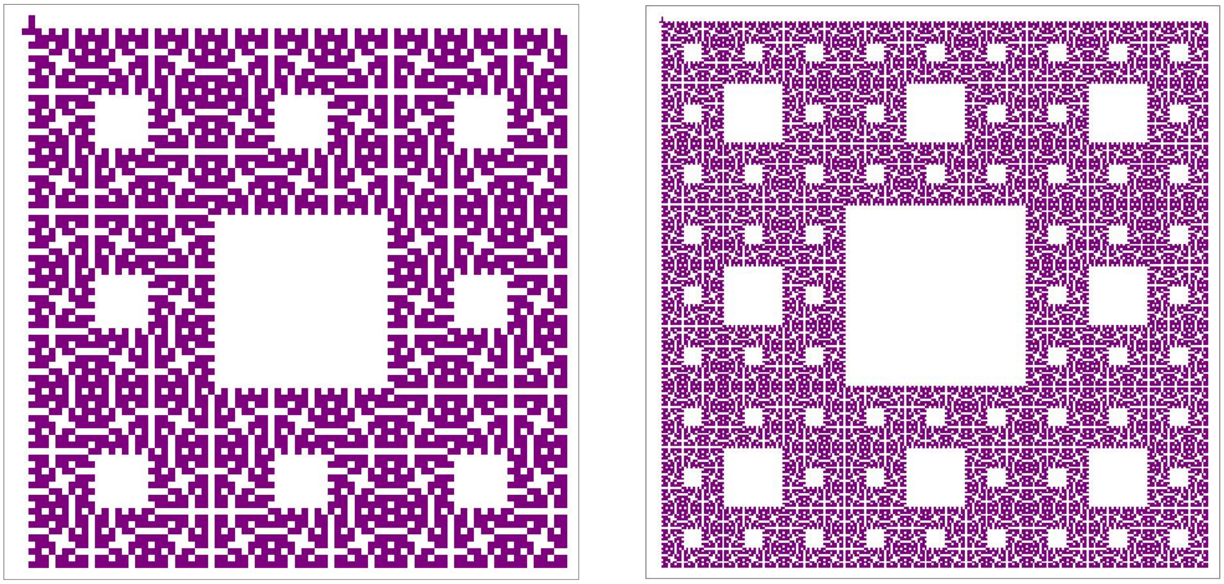


Figure 4: *Sierpinski-esque difference carpets at scales* 81×81 *and* 243×243.

We don't know where this Sierpinskiosity is coming from, much less how to prove it. Indeed, it's unclear how to describe precisely the pattern in Figure 4; there's more going on than a simple removal of middle ninths. We do know that the linearity of the formula for the colors $c_{i,j}$ means that the propagation of changes is the same for any fringe that's tweaked in the same way. Even if the fringe is colored randomly, for which we expect—and get—a random-looking carpet, adding 1 mod 3 to the first warp and weft thread, while still producing a random-looking carpet, does so with the same Sierpinski-esque difference, as shown at scale 81×81 in Figure 5 below.

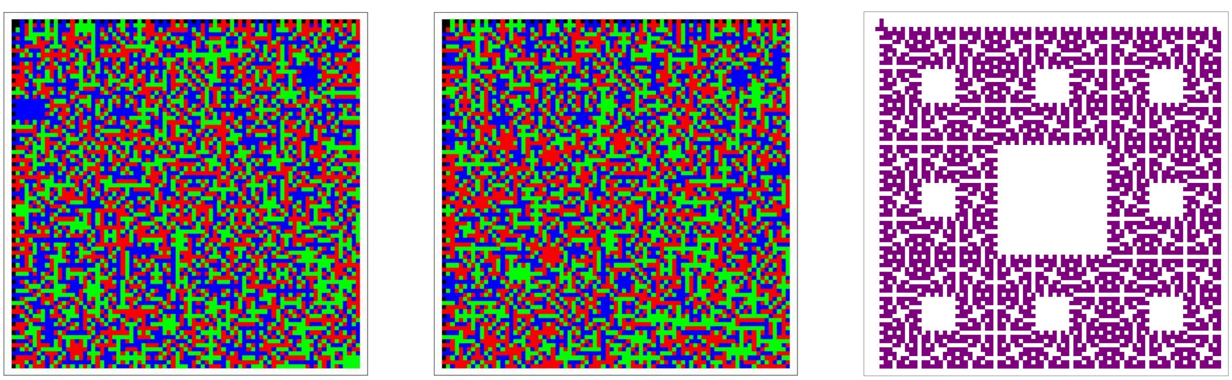


Figure 5: *A random fringe produces a random carpet (left). Tweaking the first warp and weft threads leaves the carpet looking random (middle), but the "difference" carpet is the same as in Figure 4.*

We have tried other experiments as well, such as changing just one fringe element instead of two, with similar Sierpinski-esque results. Much remains to be explored, both artistically and mathematically.

References

[1] C. Adams. *The Knot Book*. W.H. Freeman, 1994.

[2] J. Albers. *Interaction of Color*. Yale University Press, 1963.