Knit Knots: Large-Scale Soft Conformations of Minimum-Ropelength Knots and Links

Laura Taalman

James Madison University; taalmala@jmu.edu

Abstract

In this paper we present a method for constructing large, stuffed, machine-knit models of knots and links. Our approach is based on the minimum ropelength of knots, a measure of the minimum amount of rope needed to tie a given knot. We combine known mathematical bounds for minimum ropelength with physical measurements and knit gauge to determine appropriate row counts for creating large-scale stuffed models of particular knots and links on a flatbed knitting machine. The resulting soft sculptures invite tactile, playful exploration of knotted and linked mathematical forms, and are surprisingly huggable.

Motivation and Methodology

The theoretical mathematical study of knots has been an active area of research for the past century. For much, much longer, artists have incorporated motifs and forms of knots into their work; for example in the Book of Kells, Chinese knot tying, and Islamic art. Artists typically create knots using drawings, carvings, and rope, and mathematicians illustrate knots with projection diagrams, wire models, and more recently, 3D printing. We would like to build larger knot models using stuffed knit tubes. The stretchable nature of knitting will allow the material to follow the curvature of the knot. However, it is significantly time-consuming to knit at scale, especially for knot models which pack a lot of length into a small volume. We will rely on the production capacity of domestic flatbed knitting machines to produce knit tubes for this work.

Loosely speaking, the ***ropelength*** of a knot is a measure of the length of the knot when traced out by a tube of unit radius. For a tubular knot with a non-unit radius, the ropelength is the length of the knot divided by its radius [3]. The idea of this work is to compare the measurements of a simple prototype knit knot to its known ropelength upper bound, and then apply that knowledge in reverse to establish a method for determining row counts for more complex knots and links. Our goal is to provide a repeatable method for creating large-scale soft knit sculptures of knots in approximately "ideal" minimum-ropelength conformations such as those shown in Figure 1 (images from Wikipedia commons and [4]).

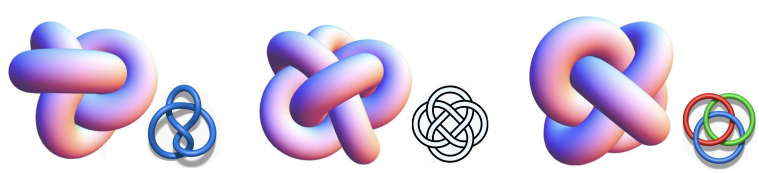

Figure 1: *Ideal minimum-ropelength conformations of the figure-eight knot, the knot* 8_{18}*, and the Borromean rings, respectively, together with their standard mathematical diagrams.*

From Ropelength to Row Count

Upper ropelength bounds for low-crossing knots and links are already known in the mathematical literature. For example, Rawdon [5] includes computed ropelength upper bounds for knots through 9 crossings showing that the ropelength of the figure-eight knot 4_1 is bounded above by 42.33, and that the ropelength of the knot known as 8_{18} is bounded above by 75.44. In Cantarella [3] the three-component Borromean rings known as 6^3_2 are illustrated with a ropelength of 58.05. Slightly sharper upper bounds for these knots can also be found in [1] and [2]. Mathematically proving these bounds is a very hard problem that we will not tackle here. In this work we will utilize known ropelength bounds to approximate rough construction lengths for our soft knot sculptures, but will not be concerned with precise minimality.

As an initial prototype we machine knitted and stuffed the figure-eight knot; see the photograph in Figure 2. This prototype was used as the basis for all physical measurements and the standard for future knit knots. The knit tube was 39 stitches around and 314 rows long, with a knit gauge of 8 stitches and 11 rows per 2”x2” square, un-stuffed tube circumference of $39 \cdot 2/8 = 9.75$”, and un-stuffed flat length of $314 \cdot 2/11 = 57$”. Note that nearly five feet of knit tubing was needed to construct even this simple knot!

Figure 2: *Flat tube knitting sample and completed figure-eight knit knot, banana for scale.*

Of course, the gauge and length measurements changed after the tube was stuffed with Poly-fil fiber and bent into the curve of the figure-eight knot. The stuffed circumference averaged about 10.5”, yielding an average ***stuffed radius*** of $10.5/(2\pi) = 1.67$”. The length of the stuffed curve was more difficult to measure, for two reasons. First, it is not possible to measure along the center of the tubular path. Second, stitches stretch out along the outside curves of the stuffed knot and become compactified along the insides of the curves. By measuring along the least stretched or compacted stitches at the “sides” of the stuffed tubes, we estimated a ***stuffed row-gauge ratio*** of 8.5 rows per 2”, and thus a length of $314 \cdot 2/8.5 = 73.88$”.

With these measurements we can approximate the “soft ropelength” of the figure-eight knot to be $73.88/1.67 = 44.24$. Note that this is *very close* to Rawdon’s upper bound of 42.33. In fact, this is a scaleup of just 4.5 percent over Rawdon’s value. If we assume that this is a consistent ***relaxing factor*** of 1.045 across knot types, then we can take ropelength upper bounds for more complicated knots and run the calculations above in reverse, thereby estimating the number of rows needed to create long enough tube to successfully form ideal stuffed models of those knots. By increasing or decreasing the relaxing factor we can also create tighter or looser knots. We can summarize this process in the following straightforward theorem.

Theorem 1. *Suppose K is a knot with an upper ropelength bound R. Furthermore, suppose that machine, yarn, and stuffing requirements result in stuffed radius r and stuffed row-gauge ratio ρ. Then the number of knit rows $\mathscr{R}$ needed to construct an ideal stuffed knot conformation of K with relaxing factor $\mathfrak{r}$ is*

$$\mathscr{R} = R \cdot r \cdot \mathfrak{r} \cdot \rho.$$

For example, our figure-eight knit knot prototype calculations had relaxing factor $\mathfrak{r} = 1.045$, stuffed radius $r = 1.67$”, and stuffed row-gauge ratio $\rho = 8.5/2$. With the ropelength upper bound from [5], Theorem 1 correctly predicts

$$44.24 \cdot 1.67 \cdot 1.045 \cdot (8.5/2) \approx 314 \text{ rows}.$$

Now let's look at another knot and see what Theorem 1 predicts. The knot 8_{18} shown in Figure 1 has a known ropelength upper bound of 75.44 in [5]. Repeating the same values for relaxing factor, stuffed radius, and stuffed row-gauge ratio, we created the "knit knot" model of 8_{18} shown in Figure 3 from a long knit tube with

$$75.44 \cdot 1.67 \cdot 1.045 \cdot (8.5/2) \approx 560 \text{ rows.}$$

Notice that to repeat the same style of knit knots as our prototype with the same machine, needles, and yarn (Loops & Threads "Impeccable Speckle"), we will always have the same $\mathfrak{r}$, r, and ρ, and therefore we can just multiply any ropelength bound R by $1.67 \cdot 1.045 \cdot (8.5/2) \approx 7.417$ to obtain an ideal row count $\mathscr{R}$. The purpose of Theorem 1 is to generalize the process for other yarns, machines, and styles.

Figure 3: *Work in progress and completed* 8_{18} *knit knot, banana and figure-eight knot for scale.*

How to Make Your Own

Hand knitting 314 rows, or worse, 560 rows, takes a long time. In contrast, each row takes only moments to knit on a knitting machine...if the machine is operational and working. There are myriad issues to keep track of while machine knitting, such as yarn tension, weight distribution, ribber bed racking and alignment, needle condition, carriage lubrication, tension mast setup, etc. If any one of these issues causes a problem then the carriage could unexpectedly catch, stitches could be lost, or the work could fall off the machine entirely in the middle of a row. The works in this paper were created using vintage equipment from the 1980s, namely a Brother KH-260 knitting machine with KR-260 ribber bed. A lot goes wrong with these machines. But, once you get the stars to align, the flat tubes knit up much faster than one could produce with hand knitting.

On a flatbed knitting machine the active stitches of the work are held on hooked needles, one for each stitch; yarn is fed through a carriage which knits each stitch as it passes from one side to the other. A second bed called a "ribber" allows for circular knitting of flat tubes, such as we need for this project. Figure 4 illustrates the knitting machine process in action and the resulting flat tubes being stitched together by hand.

Swatching. As every knitter knows, swatching is not optional. Whether knitting by hand or by machine, you should knit a short sample tube to obtain measurements for stuffed radius r and stuffed row-gauge ratio ρ, and then apply Theorem 1 with your desired relaxing factor $\mathfrak{r}$ to obtain the row count $\mathscr{R}$.

Hand knitting instructions. Cast on your preferred number of stitches onto circular or double pointed needles in waste yarn. Join, switch to main yarn, and knit $\mathscr{R}$ rows in the round. Find somewhere comfortable to do this and maybe something to binge-watch because it is going to take a long time! Switch to waste yarn and knit one or two rounds; no need to cast off.

Machine knitting instructions. On a double-bed machine, set to pitch H3 and pull out some number n of main bed needles and $n - 1$ on the ribber. Cast on in waste yarn, hang comb and weights, and switch to main

Figure 4: *Knitting the work on a Brother KH-260 knitting machine with KR-260 ribber bed, and grafting separate pieces together*

yarn and desired tension. Set main bed to slip L and ribber to slip R, and knit in the round for $\mathscr{R}$ rounds or until a problem happens and the knitting falls off or jams unexpectedly. If you make it to the end then knit a few rows of waste yarn before dropping the work from the machine. For more detailed instructions see [6].

Finishing. If you are knitting by machine, then due to various disasters you may in fact have produced a lot of shorter tubes instead of one long tube; graft these together with kitchener stitch before continuing (see Figure 4). Stuff the knit tube with filling, form into the desired shape (such as one of the minimized knot conformations shown in [4]), graft ends together, and weave in ends.

Summary and Conclusions

Viewing a small diagram or physical model of a knot can be limiting in terms of fully grasping its complexity and beauty. By creating large-scale soft sculptures, we are able to invite people to physically engage with mathematical knots and links. The friendly, huggable nature of knit knots invites exploration and play, encouraging deeper engagement with the mathematical structure of knots and links. This current work will extend to include stuffed knit models of the Borromean rings and other interesting knots and links. Future work may include (1) using the patterning capabilities of punch card knitting machines to create stuffed knit torus models with embedded illustrations of torus knots and (2) producing even larger knot models to encourage interactive play and exploration.

References

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