Kagome from Deltahedra

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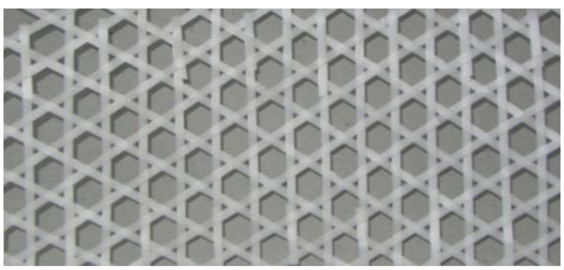
Abstract

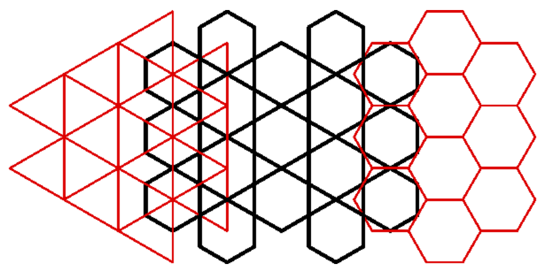
Hexagons are an obvious shape in the best known trixial basket structure, *kagome*, which is why it is often called hexagonal open weave, but actually triangles are more fundamental to its structure. This suggests the use of deltahedra as patterns for a wide variety of baskets, some having no hexagons at all. Some of them are closely related to Turk's head knots and relationships to other fibre arts, knotting and braiding, are discussed. Some new decorative knots are described.

Introduction

The commonly used term *hexagonal open weave* (or simply *hexweave*) is an accurate description of the most well-known triaxial basketry structure (Figure 1(a)) provided that the fabric is entirely flat. In practice a flat "basket" might have some limited applications, for example supported on a frame as a screen, but more usually "basket" implies some kind of container. To make one, sections of the flat structure are removed by introducing *corners*, where a strand is left out in some of the hexagons, resulting in regions that are more or less conical. At these points the weave is not hexagonal, so the Japanese term *kagome*, which simply means a basket with holes, is more accurate. Creating corners has no effect on the triangles, which are essential to the integrity of the structure.



(a)



(b)

Figure 1: *(a) A section of hexagonal open weave, (b) the structure as a rectified tiling.*

Although practical baskets generally have corners with one strand missing, creating a point of local 5-fold symmetry, it is possible to leave out two strands to have a point of local 4-fold symmetry, or even three strands, so there is a point of local 3-fold symmetry in addition to the existing triangles. The term "corner" suggests that these points could be considered to be vertices, although the weave actually looks rather different. For example, although the flat structure in Figure 1(a) will be considered as a tiling of triangles with 6-valent vertices, the physical vertices are at the mid-points of the edges of such a tiling so that both the faces and the vertices of this conceptual tiling correspond with triangles and hexagons in the actual weave. Technically the basket structure is a *rectified* triangular (or hexagonal) tiling (Figure 1(b)). Nevertheless it will be convenient to think of these structures as having triangular faces with vertices corresponding with the corners, even in flat regions when they are 6-valent and not usually considered to be corners.

Deltahedra

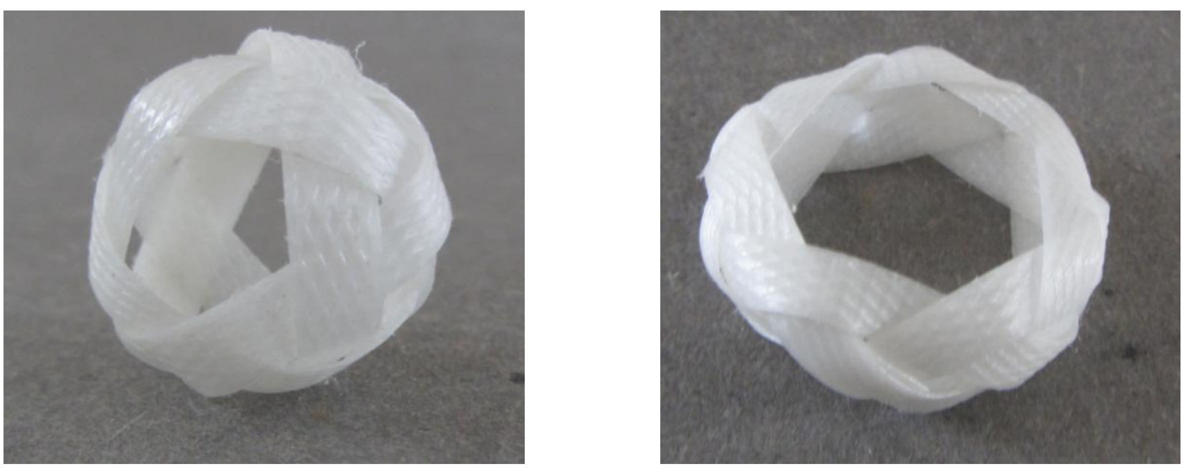
There are eight strictly convex polyhedra with equilateral triangular faces [3], [4]. Three are regular: the tetrahedron, octahedron and icosahedron. The others are generally now known by the names coined by

Norman Johnson [8]: the triangular and pentagonal bipyramids, the gyroelongated square bipyramid (a square antiprism with pyramids on each square face), the triaugmented triangular prism (a triangular prism with pyramids on each square face), and the snub disphenoid. Baskets based on these polyhedra can be made with corners corresponding to their vertices, so in the simplest cases there are no hexagons, which are excluded from the list of deltahedra by the requirement for strict convexity. Of course larger basket versions can be created by adding hexagons to make bigger faces.

The rectified forms of the regular cases (tetrahedron, octahedron and icosahedron) are the octahedron, cuboctahedron and icosidodecahedron, and they have their edges lying in equatorial planes, so the baskets derived from them consist of interlocking planar rings [7]. Although most basketry structures have strands that follow more convoluted paths, and there is enough friction to keep things in place, in these regular examples there are fewer crossings so there is a tendency for the structure to come apart. Nevertheless an icosidodecahedral basket made from rattan is stable enough to be used for a game which is popular throughout Southeast Asia, known as *takraw* in Thailand.

Bipyramids

A basket corresponding to the triangular bipyramid is very small with few crossings, and it is not very stable. The one based on the pentagonal bipyramid (Figure 2(a)), while small, holds together very well. It consists of a single strand along a closed path, so it qualifies as a knot in the mathematical sense. In fact it is an example of a "Turk's head" knot. In the classification used by the knot-tying community it has four "leads", which means that going across the knot you can count four strands; and five "bights", which means there are five strands going around the top or bottom of knot: the 5-valent corners. There is no limit to the number of leads in a Turk's head knot, so it is essentially a cylindrical biaxial structure, and the triangles occur only at the top and bottom where the strand changes direction. The flat biaxial crossings are seen as flat from the usual biaxial knot point of view, but as kagome they are 4-valent corners with positive Gaussian curvature, like a sphere. This explains why something like an oblate spheroid is produced. It happens because the strands are flat straps rather than round cords, and the overall shape is constrained by the way three strand edges touch in the triangles. The triangular bipyramid is something like a prolate spheroid since, unlike 5-valent corners, the 3-valent corners are tighter than 4-valent ones.



(a) (b)

Figure 2: *Kagome structures that correspond to Turk's head knots: (a) derived from a pentagonal bipyramid, 4L5B, (b) taking things further, 4L7B.*

In general a Turk's head knot exists whenever the numbers of leads and bights are relatively prime, so those with four leads exist with any odd number of bights. If the number of bights is even there will be either two or four separate strands. The next one after three and five bights has seven bights, abbreviated to 4L7B. A vertex with seven equilateral triangles corresponds to a surface with negative Gaussian curvature, so there is no corresponding deltahedron with regular faces. The triangles in the kagome version (Figure 2(b)) are clearly not equilateral but the 4-valent corners still work in the same way and the overall shape has a pronounced toroidal appearance. The same effect can sometimes be seen in conventional Turk's head knots, especially if they have not been tied around a cylinder.

Other Convex Deltahedra

The baskets corresponding to the bipyramids are exceptional in being mathematical knots, that is they consist of a single closed strand. The remaining three cases, the gyroelongated square bipyramid, the snub disphenoid, and the triaugmented triangular prism, consist of two, three and four strands respectively.

The basket that corresponds to the gyroelongated square bipyramid (Figure 3(d)) has eight 5-valent corners in two rings, in addition to the two 4-valent corners that correspond to the apices of the square pyramids. Such arrangements, which derive from the vertices of an antiprism, always include an equatorial strand (blue in the figure). The *takraw* ball provides a more familiar example: an icosahedron is a gyroelongated pentagonal bipyramid, a pentagonal antiprism with pentagonal pyramids on its pentagonal faces (in six ways), so it consists of six equatorial strands. If the equatorial ring is removed from these baskets the alternating over/under sequence at the crossings must be reversed in one half of the structure, which then becomes a five lead Turk's head knot, provided that the number of bights is not a multiple of five when there are five separate strands.

The snub disphenoid has four 4-valent vertices in pairs that define two edges, rather like the opposite edges of a tetrahedron. The four 5-valent vertices have the opposite orientation to those in the gyroelongated square bipyramid, so the corresponding basket has two strands (blue) along meridians, rather than a single equatorial one (Figure 3(b)).

The triaugmented triangular prism has a series of 4-valent vertices arranged equatorially. This generally results in doubled equatorial strand(s) that alternate across the equator (white). In this case it is a single strand since there is an odd number of 4-valent corners. Three more simple rings complete the basket (Figure 3(c)).

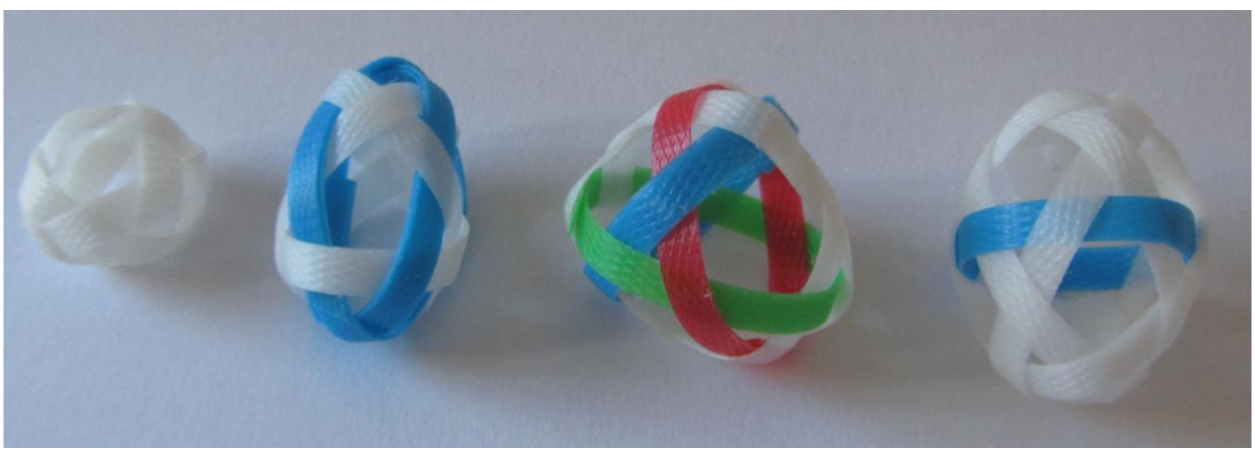


Figure 3: *Kagome from convex deltahedra: from left to right, pentagonal bipyramid; snub disphenoid; triaugmented triangular prism; gyroelongated square dipyramid.*

Adding Hexagons

Seen as an element in a deltahedron, a hexagon is equivalent to six faces around a 6-valent vertex. It is flat so it makes no contribution to the total angular deficit of a polyhedron, which must be 720° by Descartes' theorem, so there is no limit to the number of hexagons that can be added. In the simplest examples systematically adding the same number of hexagons to each face in a basket just expands the triangles to make bigger faces, which could be skewed relative to the polyhedral edges [5]. If every way of adding hexagons is considered, the number of different structures increases rapidly with the number of hexagons, and soon becomes unmanageable.

Fullerenes

Since they were discovered in 1985 there has been extensive research into fullerenes, in which carbon atoms are effectively 3-valent vertices in a range of different geometric structures. Generally the spherical examples have twelve pentagonal faces, although other possibilities are known, for example with some heptagonal faces. They are the duals of deltahedra derived from the icosahedron by adding hexagons, so the published lists of isomers [15] can be used as sources of patterns to create baskets. They are classified according to the number of carbon atoms, n. If there are n 3-valent vertices and twelve pentagonal faces, applying Euler's formula, F + V = E + 2, the number of hexagons is $3n/2 - n - 10 = ½(n - 20)$. As patterns for baskets the hexagons and pentagons become the corners of the previous discussion.

There is no polyhedron with twelve pentagons and a single hexagon, but polyhedra exist with two or more. There is only one with two hexagons, which is the dual of the next in the series: gyroelongated square bipyramid, icosahedron, gyroelongated hexagonal bipyramid, but of course a hexagonal pyramid is flat, so it is the same as a hexagonal antiprism. The basket (Figure 4(a)) has an equatorial strand (red) as discussed previously, so only one more (white) is needed, since six is relatively prime to five.

There is also only one structure with three hexagons (Figure 4(b)). They are arranged equatorially, and the basket has three strands (blue) more or less perpendicular to the equator, with a single strand (white) completing the structure, rather like Figure 3(c).

There are two isomers with four hexagons. One is the dual of the truncated tetrahedron, corresponding to a basket (Figure 4(d)) which breaks into strands forming simple rings, as might be expected given the high symmetry. Three (red) form the Borromean rings of the kagome tetrahedron. There are four more (black) in planes parallel to the hexagonal faces. The other (Figure 4(c)) does not relate in an obvious way to any structure previously considered.

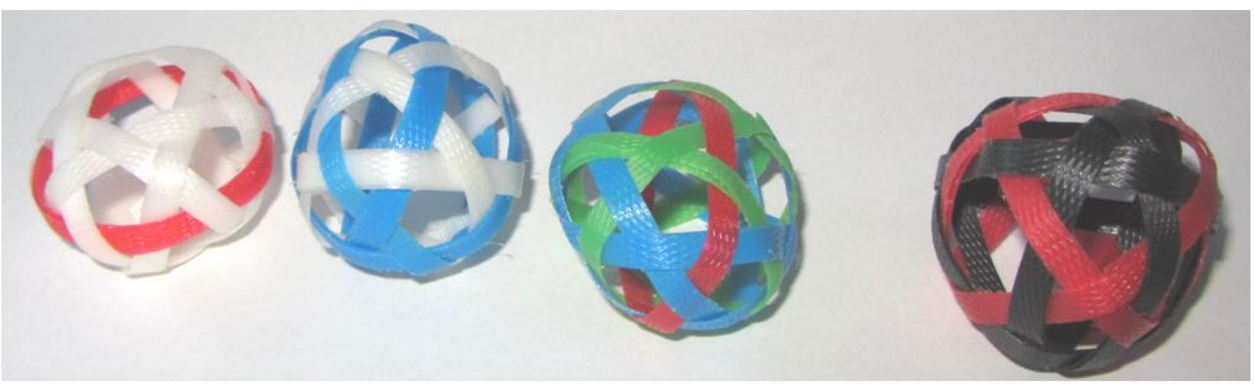


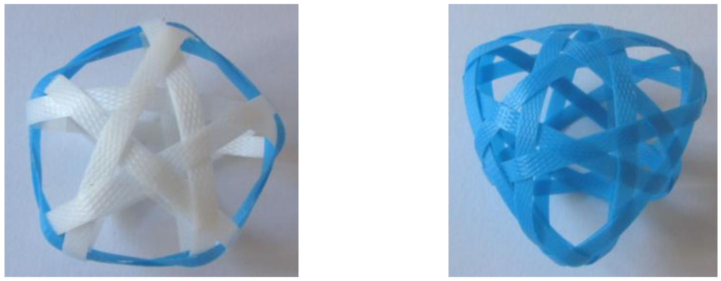
Figure 4: *Baskets based on the first few fullerenes: from left to right, with two hexagons, corresponding to a hexagonal antiprism; with three hexagons; with four hexagons; the other isomer with four hexagons, corresponding to a truncated tetrahedron.*

The number of isomers increases quite rapidly with more hexagons: there are three isomers with five; six isomers with six, and also with seven; 15 with eight; and so on, increasing approximately as n^9 [14].

Some Other Examples

Rather than systematically considering all the possible structures with a small number of added hexagons an alternative approach is to replace each triangular face with a hexagon, which retains the general overall shape. Applying this to the icosahedron generates the fourth structure that consists entirely of equatorial rings, equivalent to the edges of the dodecadodecahedron [7].

Generally adding hexagons like this can be expected to increase the number of strands. For example the 4-valent corners from Figure 2(a) now create a doubled equatorial strand, as in Figure 3(c), and the rest of the structure breaks into five simple rings, so the enlarged form (Figure 5(a)) is far from being a knot. In at least one case, however, the opposite happens. Replacing the triangles in Figure 3(c) by hexagons creates a new enlarged form that consists of a single strand. This is the only example of a knot with an obvious kagome character that I know.

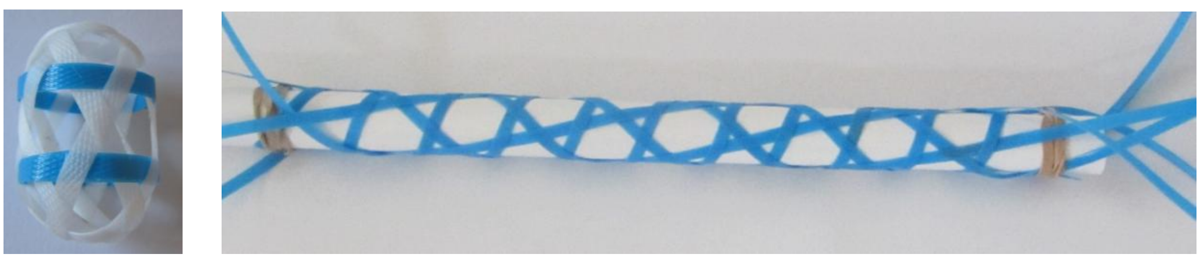


(a) (b)

Figure 5: *The effect of adding hexagons can be unpredictable: (a) the knot from the pentagonal bipyramid, 4L5B, breaks into six strands, (b) the triaugmented triangular prism becomes a knot.*

Cylinders

The simplest way to make a cylindrical deltahedron is to pile antiprisms vertically on top of each other. The corresponding basket is flat hexagonal weave rolled into a cylinder and joined at the seam. Usually there is a problem terminating potentially infinite baskets but in this case, as usual (compare with Figures 3(d) and 4(a)), the vertices of the top and bottom antiprisms correspond to 5-valent corners and there are no loose ends. The horizontal strands are simple rings (Figure 6(a)). If they are removed, and the necessary over/under adjustments made, a Turk's head knot is left, in this example 6L5B.



(b)

Figure 6: *Cylindrical kagome (a) from a stack of three pentagonal antiprisms, (b) from the tetrahelix.*

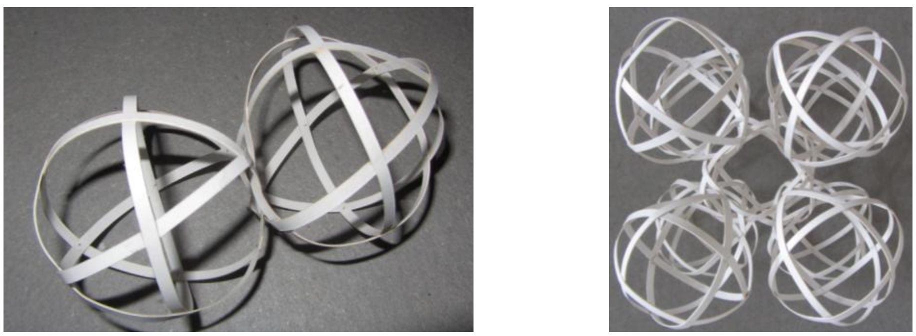
If the seam edges are displaced along the seam before being joined, the strands follow helical paths. The minimal case corresponds with the tetrahelix, formed when tetrahedra are stacked together. There is a single strand that winds around the cylinder in one of the three triaxial directions. Of the two other directions, one has two strands winding in the opposite sense, and the other has three strands winding in the same sense (Figure 6(b)). Of course there are larger structures that correspond to cylindrical polyhedra composed of helicoidal strips of equilateral triangles [16]. There is no tidy way to close the ends of these cylinders.

Triply Periodic Infinite Polyhedra

Some people have used basketry to model triply periodic surfaces. At Bridges in 2005 Richard Ahrens exhibited three examples that consist of kagome versions of polyhedra joined together [1]. Alison Grace Martin has explored many more possibilities that use kagome to model surfaces of negative curvature. At Bridges in 2013 she exhibited a version of the Schwarz P surface [9].

Negative curvature implies corners with valency > 6, but they become increasingly difficult to work with as the number of strands increases, so it is easier to weave structures that have many hexagons so the curvature is spread over larger areas. It is then possible to use heptagons alone to introduce curvature. Minimal examples without hexagons, comparable to those in Figure 2, present particular difficulties.

A polyhedral version of the P surface can be made by removing the square faces from a packing of truncated octahedra. It can be a pattern for kagome since it consists only of hexagons. There are four of them at a vertex, requiring 8-valent corners. The surface divides space into congruent halves, and there are polyhedra on both sides of each hexagon, so they consist of interlaced triangles in the basket model, which consists of kagome octahedra. This avoids having to deal with the corners directly. The kagome octahedron is a effectively a spherical cuboctahedron with four interlaced equatorial rings, so it is not very stable when made from plastic strapping, but gluing strips of card provides a reasonable alternative (Figure 7(a)). A further problem arises because of a tendency for the interlaced triangles to twist. This would not happen in an infinite structure because each octahedron would be kept in place by its neighbours but in practice the ones on the outside are missing some neighbours (Figure 7(b)).

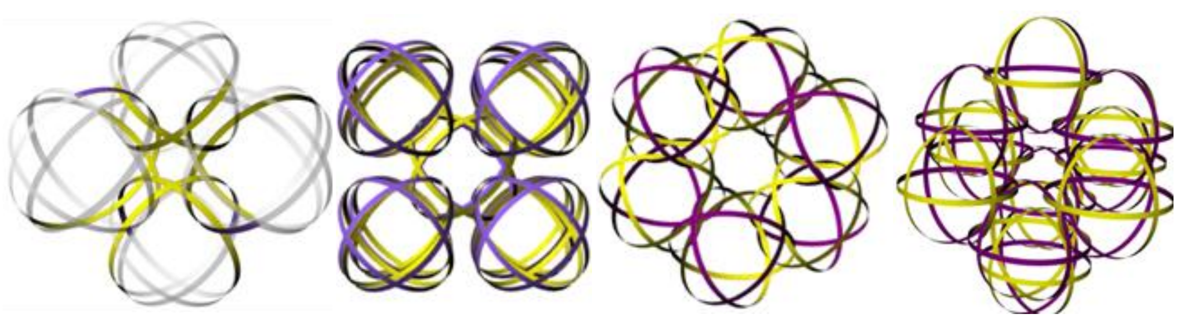


(a) (b)

Figure 7: *Steps in constructing a model of Schwarz's P surface (a) two kagome octahedra sharing a hexagon, (b) a central octahedron with all of its eight neighbours.*

It is not easy to see the 8-valent corners in a physical model, but a computer-generated image makes them more obvious (Figure 8(a)).

The same problems arise in trying to construct a minimal model of the D surface, and they are even more extreme. The truncated tetrahedra of the regular version are represented by kagome tetrahedra: Borromean rings. The corners are 12-valent (Figure 8(c)), and the twisting tendency is more pronounced. It is not really possible to make a physical model, and computer images seem to be the only way to show this structure effectively (Figure 8(d)).



(a) (b) (c) (d)

Figure 8: *Computer-generated images of minimal versions of triply periodic surfaces coloured to show the two sides (a) an 8-valent corner of the P surface, (b) a view of the P surface as in Figure 7(b), (c) a view of a 12-valent corner of the D surface, (d) a view of more polyhedra from the D surface.*

Discussion

Traditional crafts have evolved over centuries, and each has its own repertoire of forms and structures that form the basis for further developments. When ideas that are part of one tradition cross into another some interesting innovations can result. For example almost all of the traditional fibre arts implicitly assume a basic biaxial structure, probably from a focus on individual crossings, each of which is locally biaxial. Triaxial basketry is unusual since hexagons and pentagons occur naturally but are rare in biaxial structures, and cross fertilization between the traditions can inspire original developments. Even more becomes possible when more recent research from unrelated disciplines suggests further design ideas.

The discovery of a new decorative knot provides a good illustration. As long as baskets are seen only as containers, some decorative forms that are commonplace in knot-tying traditions are inconceivable. If closed forms are considered then new kinds of design become possible. Of the six fullerenes with seven hexagons there is one with 3-fold symmetry (Figure 9). Using this as a pattern leads to a closed basket that has two strands. One of them alternates around the equator, and its removal obviously leaves a single strand. Since there are 19 corners (including the hexagons) what remains is enough to be stable. Making the necessary over/under adjustments produces a knot that has three 4-valent, six 5-valent corners, and a single hexagon, so it could be derived from Figure 3(c) by adding a single hexagon.

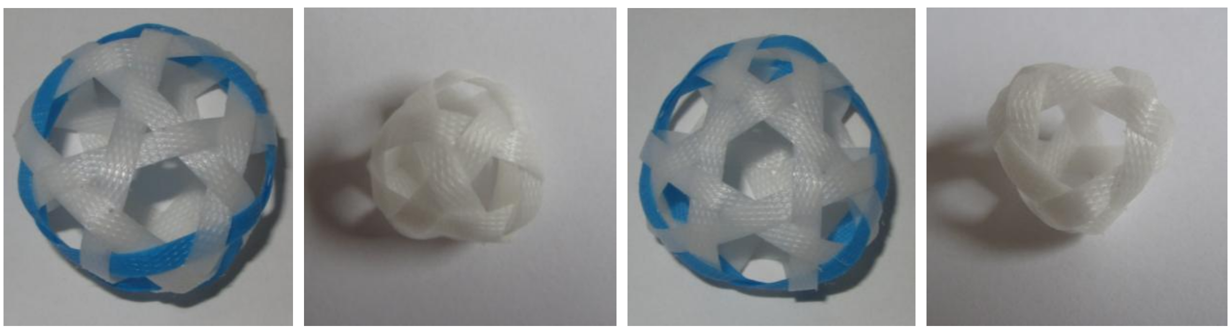


Figure 9: *Two views of a basket derived from a fullerene that has seven hexagons, along with views of the knot that results when the blue equatorial strand is removed (after adjusting the crossings).*

Methods of Construction

Kagome baskets are usually made from pieces of plants, which are generally limited in length. Construction begins by making a single hexagon (or corner) with six (or fewer) strands. More hexagons/corners are added using a braiding technique, and new pieces of strand are joined in as the construction proceeds. Such joins are unavoidable if traditional materials are used, but with plastic strapping the joined-together short sections can be replaced with a single long piece once the structure is complete. This creates a more uniform appearance, and using different colours makes it easier to distinguish the strands if there is more than one.

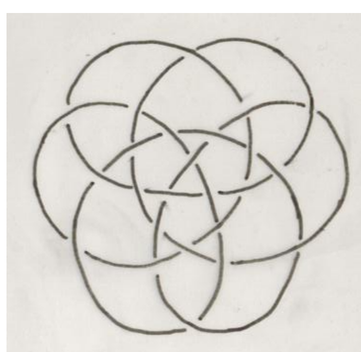


Figure 10: *A diagram for the knot in Figure 9.*

Tying a knot is entirely different. There is only a single cord right from the beginning so the details of the path must be known in advance, rather than allowing things to develop the way they do in basket-making. The standard reference book on knots [2] gives diagrams for each one, and more recent books of instruction use photographs to show each step in tying the knots they describe [11]. Figure 10 shows a diagram for the knot in Figure 9. Knotting techniques allow more freedom over the final appearance. For example a knot does not have to be polyhedral. It could be flat, looking more like the diagram, and passing the cord around the knot two or more times is a common device to improve its appearance.

Braids

Braids of various kinds can be produced using knotting techniques, but there are other braid traditions [10]. Hollow braids are known from the Mediterranean region and Japan, and their connection with basketry is recognised. They are all essentially biaxial cylinders, but I think triaxial helical forms like Figure 6(b) have never been developed. It should be possible to create them using standard braiding methods. Even madweave is possible since it can be produced starting from kagome [6], although the procedure might be quite complicated. Solid braids might also be possible following patterns provided by the recently described "star columns" [12], [13], although the dihedral angles between the equilateral triangles make them unsuitable as patterns for kagome.

Acknowledgement

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