What if Albrecht Dürer had Unfolded
Leonardo da Vinci's Elevated Polyhedra?

Rinus Roelofs

Lansinkweg 28, Hengelo, the Netherlands; rinus@rinusroelofs.nl

Abstract

At the beginning of the 16th century two books with drawings of polyhedra appeared: in 1509 *Divina Proportione*, with drawings by Leonardo da Vinci was published, and in 1525 *Unterweisung der Messung mit dem Zirkel und Richtscheit* by Albrecht Dürer was published. Dürer almost certainly knew the drawings of Leonardo da Vinci; all polyhedra from *Divina Proportione* can be found in Dürer's book, except for the elevated polyhedra, polyhedra where a pyramid has been added to each face to create star-shaped polyhedra. Is there a reason for this omission? We do not know, but in any case, this could be a good reason to see what Dürer could have produced.

Introduction

In 1525, Albrecht Dürer published the planar nets of regular and some semi-regular polyhedra [3]. That was the first time that nets of polyhedra were published and they must have revolutionized the polyhedron modeling world. Never before had it been so easy to produce, for example, a model of a dodecahedron. After all, it was no longer necessary to calculate the angles between the planes first. You could now cut the pattern out of paper and fold the pentagonal surfaces one to the other and then glue them together. In Figure 1 we see Dürer's drawings of the unfoldings of the 5 Platonic polyhedra. In addition to the flat nets, the top and front views of the final model are also always drawn.
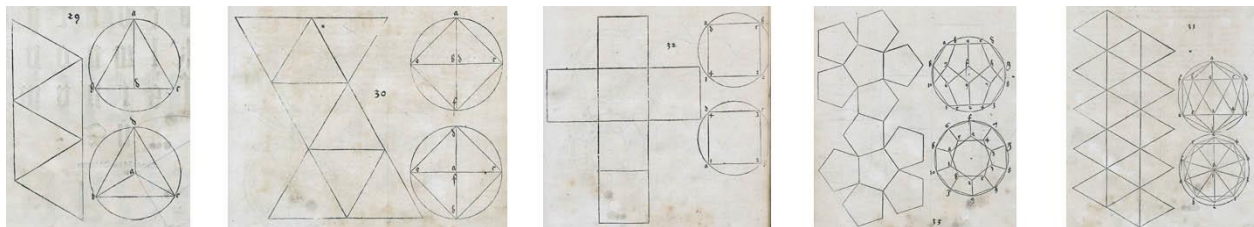

Figure 1: *Dürer's drawings of the Platonic solids.*

Luca Pacioli's book *Divina Proportione* [4], illustrated by Leonardo da Vinci, was published in 1509, 16 years before Albrecht Dürer published his book. Here too we see images of the regular and some of the semi-regular polyhedra. In Figure 2 we see Leonardo's drawings of the 5 Platonic polyhedra.
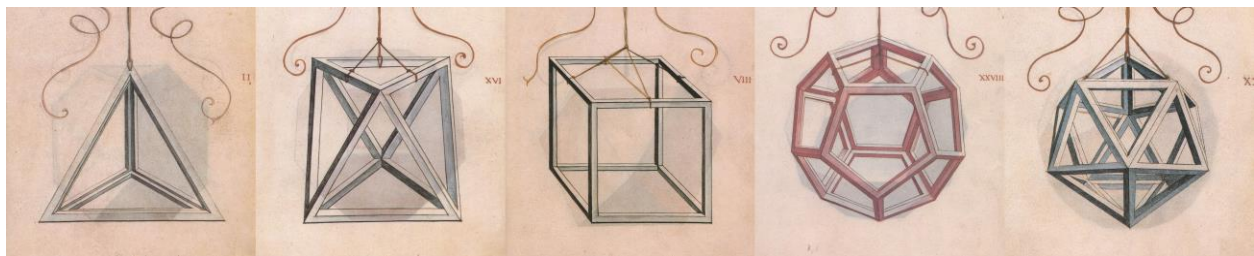

Figure 2: *Leonardo da Vinci's drawings of the Platonic solids.*

We encounter all the semi-regular polyhedra of Leonardo da Vinci in Albrecht Dürer's book, and even three more. The question arises whether Dürer knew Leonardo's drawings. We are fairly certain that this is the case [1] and that Dürer at least had the book by Pacioli and Leonardo in his hands. Is there a reason why Dürer did not make any flat nets of the elevated polyhedra? Of course, there could have been very plausible reasons, such as lack of time. Albrecht Dürer died in 1528. Maybe he didn't find it interesting enough, or not essential enough. He may even have thought it impossible. It was not until 1568 that we saw a first publication of the nets of the elevated polyhedra in Barbaro's book *La Pratica della Perspettiva* [2].

Dürer's Drawings

Albrecht Dürer's polyhedron drawings are special in more than one way. If we look at the drawings of the regular polyhedra, we always see the drawing of the flat net, and next to it the drawings of a top view and a front view. So actually two stages of a process are depicted: from 2D to 3D. Nowadays we can also present this through an animation. The intermediate part, the movement from 2D to 3D, thus receives more attention, and even an aesthetic value. This aspect was perhaps not so strong for Albrecht Dürer, because it was difficult to record transition, although you could of course draw a number of intermediate steps as in Figure 3.
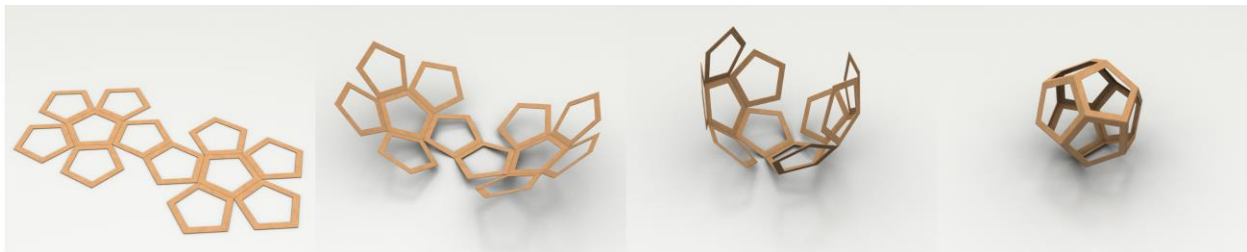

Figure 3: *Folding Dürer's net of the dodecahedron.*

A First Analysis

When we look at Dürer's 14 nets of polyhedra, certain choices stand out. Choices had to be made because the number of possible nets for the polyhedra quickly increases as the number of faces increases: from 2 nets for the tetrahedron, 11 for the cube and also the octahedron, to 43,380 for the dodecahedron. For the dodecahedron (Figure 3), for example, Dürer clearly opts for a setup of two halves. And we find this arrangement in three other cases: the octahedron, the icosidodecahedron and the truncated icosahedron (Figure 4). It seems to simplify the problem of finding a net.
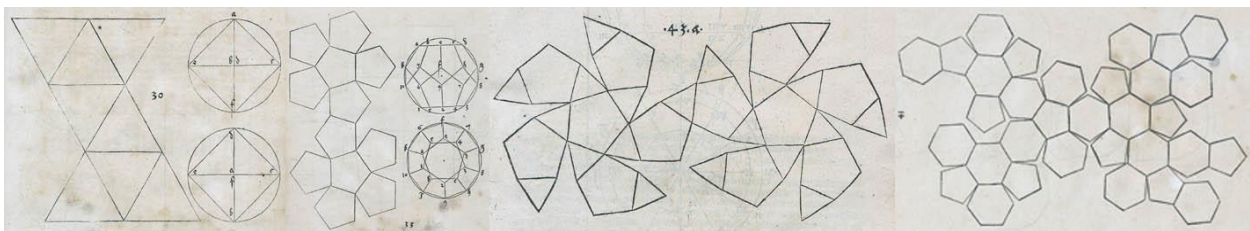

Figure 4: *Half-half layout for octahedron, dodecahedron, icosidodecahedron and truncated icosahedron.*

Another design is the Star layout. The net is set up from a chosen central face. We find this design, for example, in the net of the cuboctahedron and in three other cases: the truncated octahedron, the snub cube and the rhombitruncated cuboctahedron (Figure 5). It is a design with certainly an aesthetic value.
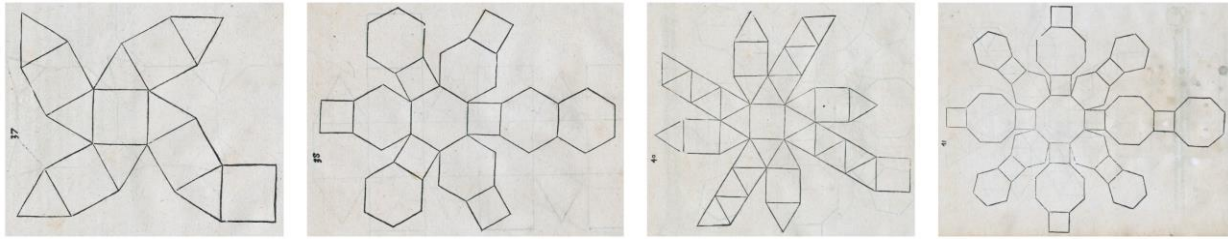
Figure 5: *Star layout of the nets. Orientation of Dürer's drawings changed to show the similarity.*

A third recognizable design is the Equator layout, seen most clearly in the net of the icosahedron. There are three other cases of this design: the cube, the truncated cube and the rhombitruncated cube (Figure 6). Maybe these designs were inspired by the way globes were made. But I couldn't find any evidence on this.
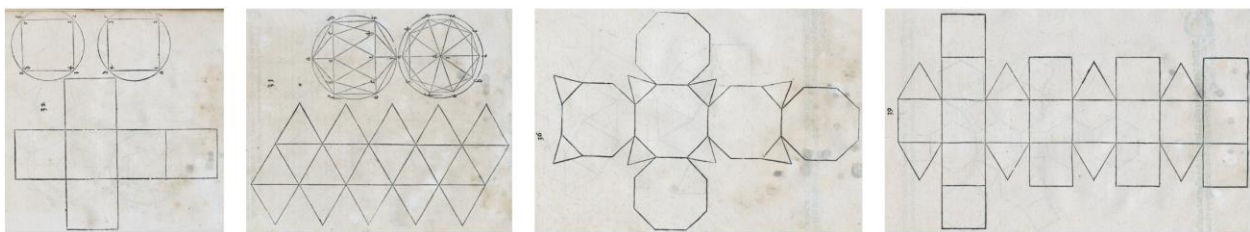

Figure 6: *Equator layout of the nets. Orientation of Dürer's drawings changed to show the similarity.*

We thus see three different strategies to arrive at a net of a polyhedron. Two of Dürer's fourteen nets do not fall into these categories: those of the tetrahedron and the truncated tetrahedron. The choice of the shape of the tetrahedron net (Figure 1, first left) is actually the most surprising. From the two possible nets, the "big triangle" was perhaps more obvious because of its star-shaped layout, and also because Dürer used two "big triangles" when constructing the net of the octahedron (see Figure 1, second left). But Dürer chose the Strip layout here, a strip of 4 triangles. And this choice is especially good for the transition process, moving from 2D to 3D. The animation of this transition is very interesting because of the beautiful movements that arise. Albrecht Dürer certainly shows a certain approach that perhaps stems from his artistry. It is now interesting to analyze how he might have dealt with the problem of finding nets for the elevated polyhedra.

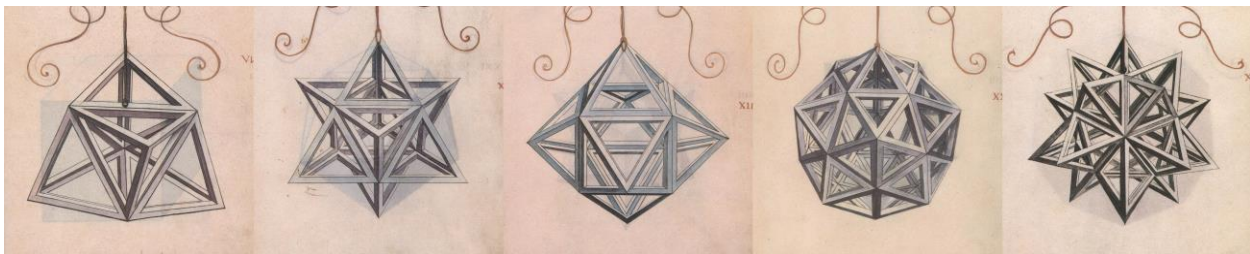## Leonardo da Vinci's Drawings of the Elevated Polyhedra

Figure 7: *Leonardo da Vinci's drawings of the elevated polyhedra.*

In the book *Divina Proportione* the elevated polyhedra are presented in two ways. Luca Pacioli described them as polyhedra with a pyramid of equilateral triangles placed on each face. The faces of the original polyhedron are still within the elevated polyhedron. In the cut-away drawings by Leonardo da Vinci,

however, it is clear that these inner surfaces have been omitted (Figure 7) by Leonardo da Vinci. With this presentation, those polyhedra meet the usual definition of polyhedra where exactly two faces are linked to each edge. For Albrecht Dürer, it would also have been most obvious to assume this. This is because we can regard his drawings as coverings of the polyhedra, so that the inner space is left out of consideration.

Unfolding the Elevated Polyhedra

We can obtain a net of the elevated tetrahedron when we unfold this polyhedron as a star-shape, following one of Albrecht Dürer's strategies. We then see that the total array of triangle faces is actually composed of three strip nets of the tetrahedron (Figure 8).
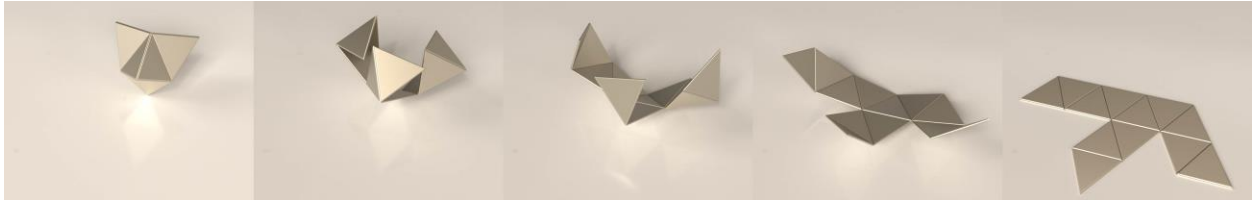

Figure 8: *Unfolding the elevated tetrahedron.*

We can also generate these star-shape solutions for the other elevated Platonic solids. Figure 9 shows how the elevated octahedron, also known as the stella octangula, can be unfolded into a net consisting of three linked strips of 8 equilateral triangles each.
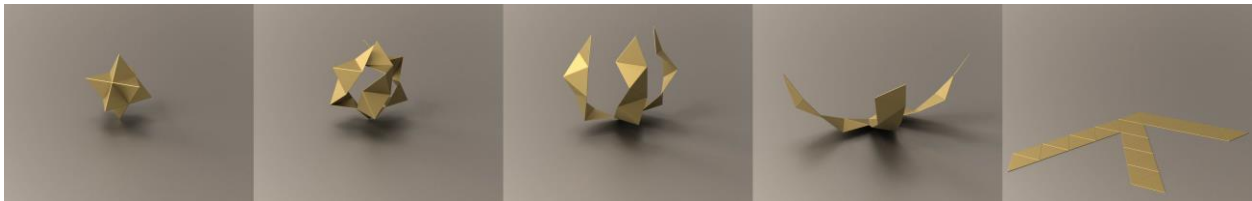

Figure 9: *Unfolding the elevated octahedron.*

And in Figure 10 we see how the unfolding of the elevated dodecahedron results in its net.

Figure 10: *Unfolding the elevated dodecahedron.*

Unfolding the Elevated Polyhedra: Two-step Solution

Another setup for creating unfoldings of the elevated polyhedra could be by splitting the trajectory of folding into two parts. We take as the "halfway point" the net of the basic polyhedron and then replace each face of this net with a pyramid. We can take Albrecht Dürer's net or any other net of the basic polyhedron. After all, the whole can then be collapsed as if it were the original polyhedron. In Figures 11 and 12 we demonstrate this two-step technique of unfolding the elevated tetrahedron. In the first step (Figure 11) we

see the unfolding of the elevated tetrahedron into a strip of four open tetrahedra in which Albrecht Dürer's original net for the tetrahedron can be recognized.

Figure 11: *Unfolding the elevated tetrahedron, first step.*

In the second part of the trajectory (Figure 12), the open tetrahedra are unfolded and laid flat to arrive at the final net.

Figure 12: *Unfolding the elevated tetrahedron, final step.*

For my own research I have given increasing importance to the beauty of animation. With this 2-step method, this has sometimes led to different choices for the half-way layout. For the cube, for example, I chose a strip pattern as the middle figure instead of Albrecht Dürer's cruciform pattern. In Figure 13 we see the first part of the unfolding of the elevated cube. And in Figure 14 we see the final net of the elevated cube.

Figure 13: *Unfolding the elevated cube, first step.*

Figure 14: *Unfolding the elevated cube, final step.*

With this 2-step method, we can develop different nets for the elevated polyhedra. First, we have the choice of the halfway layout we want to use, and then there are different ways of unfolding the open pyramids to

arrive at the final net. In the following example of unfolding the elevated tetrahedron (Figure 15), the first part of the unfolding is similar to that in Figure 11.
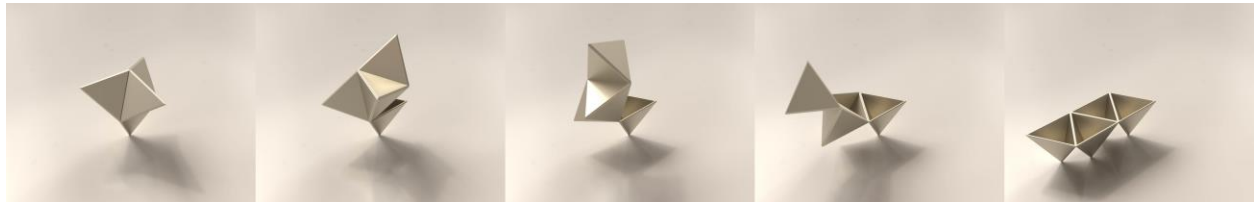

Figure 15: *Unfolding the elevated tetrahedron, first step.*

We can now unfold the four open tetrahedra so that a net in the form of a long strip of 12 equilateral triangles is created (Figure 16). This is quite different from the unfolding of Figure 12.

Figure 16: *Unfolding the elevated tetrahedron, final step.*

If we look at the first part of the 2-step solution for the elevated icosahedron, we see Albrecht Dürer's net of the icosahedron at the halfway step (Figure 17). Dürer's net of the icosahedron can be seen as a composite of five tetrahedron nets.

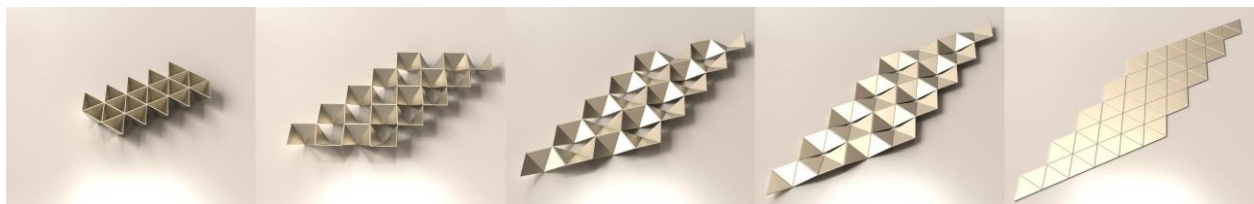**Figure 17:** *Unfolding the elevated icosahedron, first step.*

Figure 18: *Unfolding the elevated icosahedron, final step.*

In the second part of the unfolding of the elevated icosahedron (Figure 18) we see five strips of 12 triangles, that is five times the strip of four open tetrahedra seen in Figure 16 (far left) for the net of the elevated tetrahedron. The end result is a simple net and the movement from the 3D elevated icosahedron to its flat net produces a nice animation. The beauty of the animation was also paramount in my choice of unfolding

the elevated octahedron. For the first part, the net of the octahedron, I chose not the half-half layout of Albrecht Dürer, but 8 triangles in a strip shaped like an S (Figure 19).
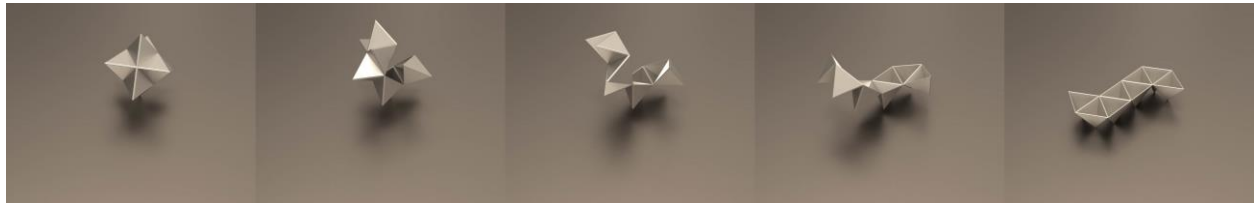

Figure 19: *Unfolding the elevated octahedron, first step.*

The unfolding in the second part (Figure 20) now gives a very nice movement, resulting in an alternative net of the elevated octahedron.

Figure 20: *Unfolding the elevated octahedron, final step.*

Equator Solutions

In the equator solution of creating the net of the elevated octahedron, the top and bottom pyramids of the elevation are already split into separate triangles in the first phase of the unfolding process (Figure 21).

Figure 21: *Unfolding the elevated octahedron, first step.*

The second part of the process (Figure 22) unfolds the strip of six open tetrahedra in the "equator," leading to a nice simple net of the elevated octahedron that can be presented by an animation.
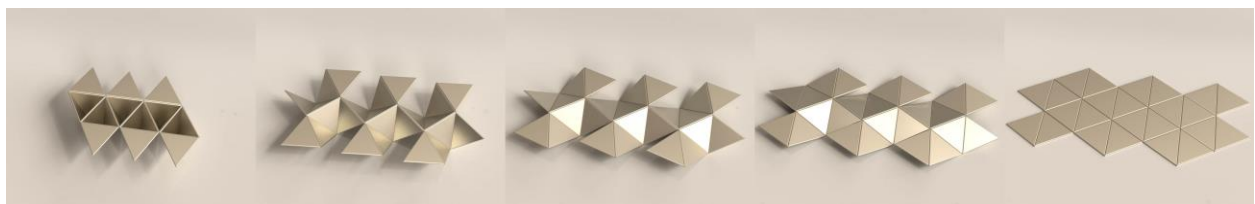

Figure 22: *Unfolding the elevated octahedron, final step.*

Hamiltonian Paths

Nets of elevated polyhedra that form of one continuous strip of triangles are very suitable for animations. There is a nice method to generate this kind of net. First, find a Hamiltonian path on the dual of the elevated polyhedron, that is, a connected path of edges that visits each vertex exactly once. This then corresponds to a simple path connecting all of the faces of the original polyhedron. In Figure 23, we show the unfolding of the elevated octahedron using this method. Here a simple path of triangular faces on the elevated octahedron unwinds to a net that is a single strip of triangles.

Figure 23: *Hamiltonian path solution of the net of the elevated octahedron.*

Figure 24 shows the Hamiltonian path solution for the elevated icosahedron. Both examples, the Hamiltonian path solution of the elevated octahedron and the Hamiltonian path solution of the elevated icosahedron, are best seen in animation.

Figure 24: *Hamiltonian path solution of the net of the elevated icosahedron.*

Conclusion

Making nets of elevated polyhedra may at first seem like a difficult process or maybe even impossible. Nonconvex polyhedra are not always unfoldable in a single net. For all of the elevated polyhedra published in Luca Pacioli's book *La Divina Proportione*, as well as for a few not-published elevated Archimedean polyhedra, it appeared to be possible to unfold them in a single net. In contrast to the nets of the Platonic and Archimedean solids, it is very difficult to recognize the elevated polyhedron by its nets. The most interesting part therefore is the animation which makes the connection between net and polyhedron.

References

[1] N. Andrews. *The Polyhedrists.* The MIT Press, 2022.

[2] D. Barbaro. *La pratica della perspettiva: opera molto vtile a pittori, a scvltori, & ad architetti: con due tauole, una de' capitoli principali, l'altra delle cose piu notabili contenute nella presente opera.* Venice: Camillo & Rutilio Borgominieri, 1568.

[3] A. Dürer. *Unterweisung der Messung mit dem Zirkel und Richtscheit in Linien, Ebenen und ganzen Körpern, durch Albrecht Dürer zusammengozogen und zu Nutz aller Kunstliebhabenden mit Zugehörigen Figuren in Druck gebracht im Jahr MDXXV.* Nurnberg, 1525 (2nd ed. 1538).

[4] L. Pacioli. *La Divina Proportione.* 1509, Ed. Akal, S.A., Madrid, 1991.