Materializing Daniele Barbaro's Creativity with 3D Printing

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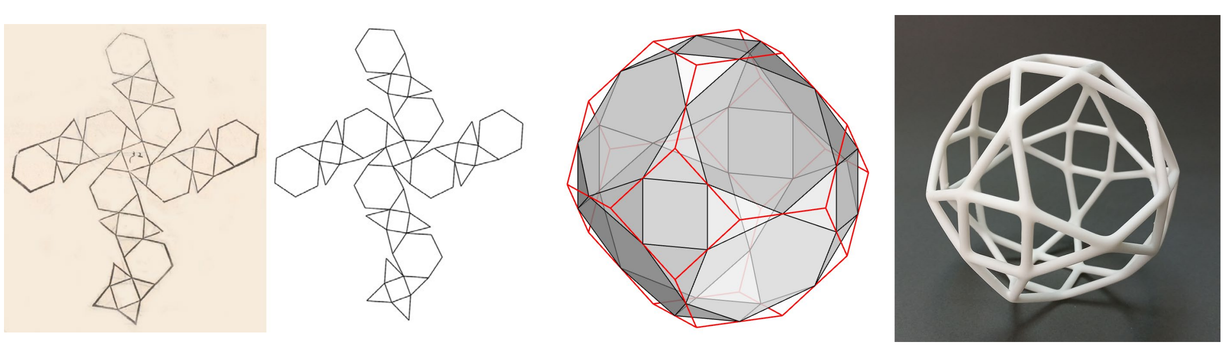
Abstract

The paper considers nine of the polyhedra that Daniele Barbaro described in his 1568 book *Pratica della Perspettiva*. Planar nets that correct geometrical errors in Barbaro's drawings and 3D-printed models as possible realizations are presented.

Polyhedra in Pratica della Perspettiva

Daniele Barbaro (1514–1570) was a Venetian humanist and architect mostly known for his writings on architecture. His last book, *Pratica della Perspettiva* [1], published in 1568, is a treatise on perspective drawing meant for painters, sculptors, and architects, as stated on pages xiii-xiv of [2]. In Part III, Barbaro describes 39 polyhedra, 11 of which are Archimedean solids, and explains how to obtain each and others derived from them. Barbaro believes that infinitely many bodies are obtainable by sequentially truncating the vertices of other bodies, as denoted on page 111 of [1]. Luca Pacioli (ca. 1447–1517) and Albrecht Dürer (1471–1528) expressed similar ideas on pages 118 of [8] and 304 of [4]. Barbaro describes 32 solid bodies with a planar net or *spiegatura*, adding, in some cases, plan-views and perspective drawings. Before him, the only authors who consistently explored polyhedral nets were Albrecht Dürer, to describe 16 solid bodies in 1538 [3], and Augustin Hirschvogel (1503–1553), to describe 10 in 1543 [5]. Barbaro shows 15 solid bodies only in *spiegatura* (some of them partial), leaving no clues as to why he did not include any other drawing; perhaps he found it unnecessary or tedious. A few planar nets show inconsistencies, which makes them difficult to understand. It is possible that these and other imprecisions occurred because Barbaro never got to craft these solid bodies as three-dimensional models, which would explain why he did not include more drawings: Barbaro could not study them with the same level of detail as when he had the chance to work with "the material body in hand," as stated on page 32 of [12]. And it would also explain why he did not understand which set of faces, in each case, could not be regular.

Barbaro's treatise was recently translated into English [2], and it will undoubtedly be a valuable source for future studies, as it has been to us. This paper summarizes some conclusions from our previous research [10] on the non-uniform polyhedra in *Pratica della Perspettiva* and aspires to make Barbaro's findings more understandable by presenting a corrected version of Barbaro's *spiegature* and propose their materialization in stereolithographic printing that we have chosen for its level of detail and accuracy.

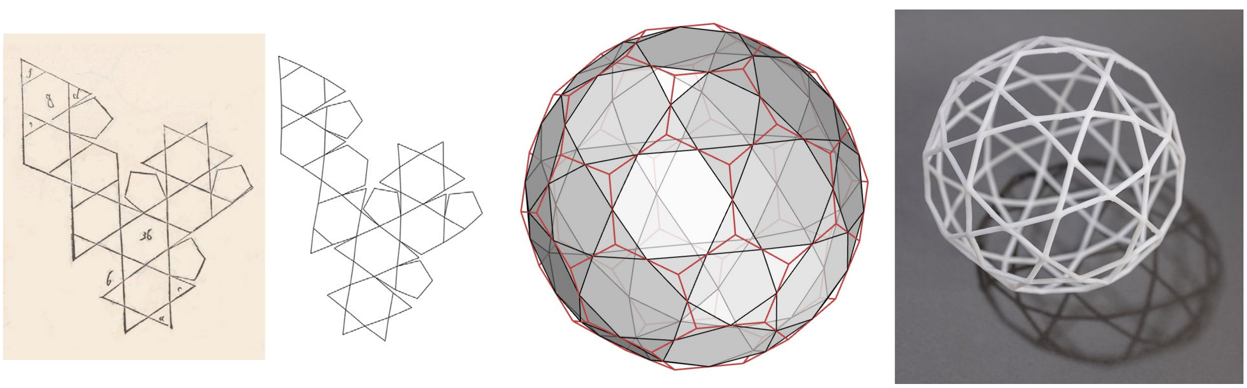


(a) (b) (c) (d)

Figure 1: *Rectified Truncated Octahedron in Chapter XVI, pages 91-93 of [1]: (a) spiegatura, (b) corrected planar net, (c) virtual model, (d) 3D print (diameter: 100 mm; pipe radius: 2.0 mm)*

Barbaro obtained the solid body in Figure 1 by dividing the edges of the body of six squares and eight hexagons into two equal parts, as stated on page 91 of [1], and shows it in planar net, plan-view, and perspective drawing. Wentzel Jamnitzer (1507/08–1585) drew a similar body on Plates B. II and B. IV of *Perspectiva Corporum Regularium* [6], published in the same year as *Pratica della Perspettiva*.

Truncating the vertices of a polyhedron after dividing its edges in half is an operation known as rectification. When we rectify an Archimedean solid, the edges' midpoints become vertices of new regular faces, but the edge lengths differ in each type of face. Since the area of these new faces is smaller, additional faces (usually a set of triangles or sets of triangles and quadrangles) fill the space between the new ones. Figure 1(c) shows that the space between the new pentagonal and hexagonal faces is filled with triangular faces that cannot be regular but isosceles. However, this does not seem evident in Barbaro's *spiegatura* in Figure 1(a), so Figure 1(b) shows a corrected planar net. Figure 1(c) is a virtual model showing how the rectified truncated octahedron derives from the truncated octahedron. In the 3D print, we deleted the faces and replaced the edges with cylinders, giving their intersections an organic feel. Long pipes tend to break during printing in models with large faces and fewer edges; this was the first model in which it happened. We solved the problem by slightly increasing the cylinders' radius, making the pipes thicker than in other prints.

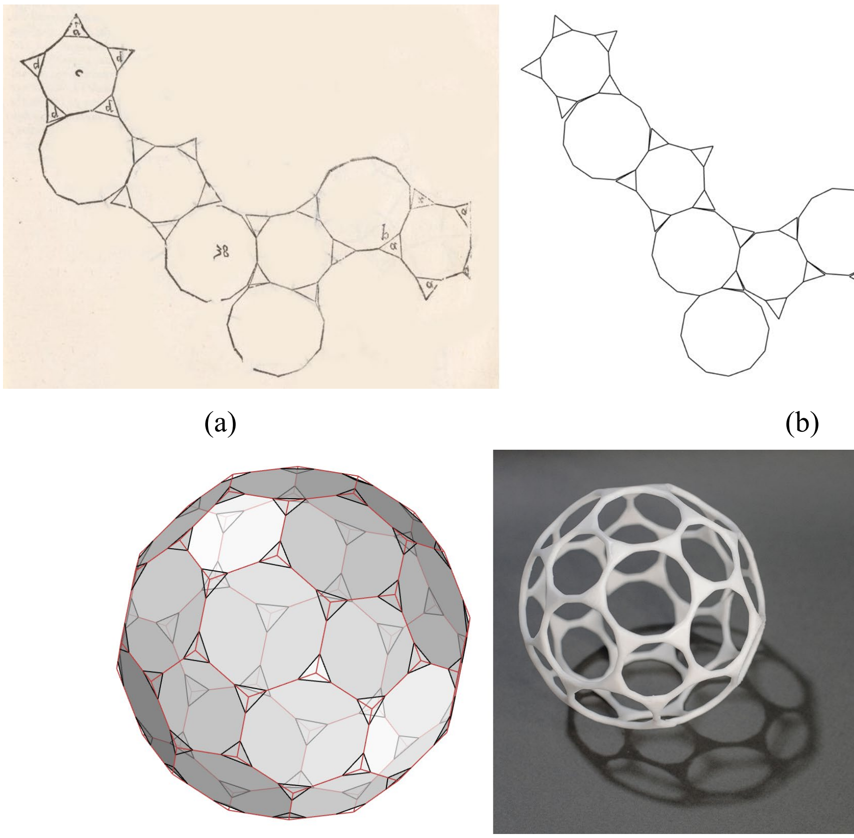


(a) (b) (c) (d)

Figure 2: *Rectified Truncated Icosahedron in Chapter XVIII, page 97 of [1]: (a) spiegatura, (b) corrected planar net, (c) virtual model, (d) 3D print (diameter: 100 mm; pipe radius: 1.0 mm)*

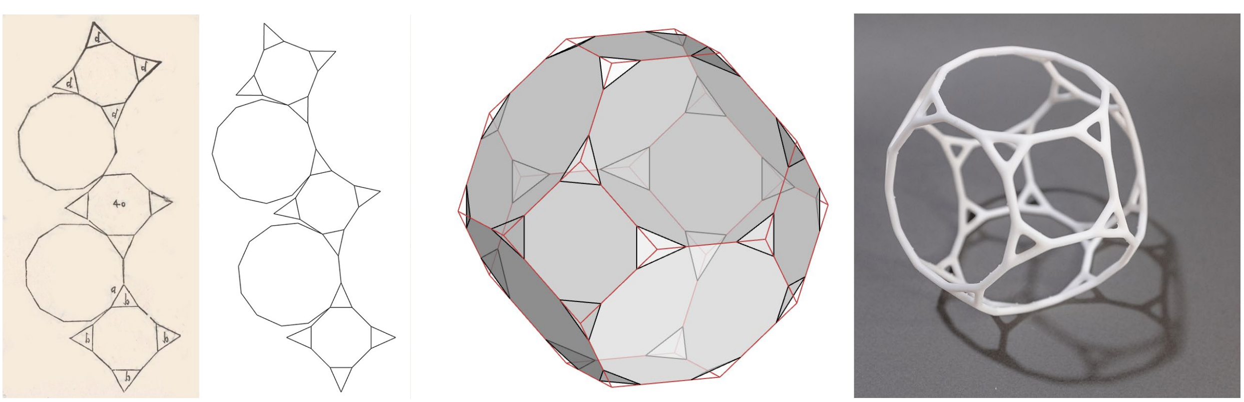
Barbaro introduces another body he obtained from the body of 20 hexagons and 12 pentagons, according to his own words on page 97 of [1]. He does not say its faces are regular or that all the edges have the same length, but the fact is that the planar net was drawn as if they were, yielding a bizarre outcome in which the vertices where hexagons and triangles intersect match a plane tessellation. The rectified truncated icosahedron in Figure 2 is a near-miss Johnson Solid because its triangular faces are close to regular. Kaplan and Hart identify the rectified truncated icosahedron and the rectified truncated octahedron as symmetrohedra on pages 27 and 21 of [7]. As to its 3D print, since this model has shorter edges than the previous model, it was possible to print it with thinner pipes.

Barbaro mistakenly says, on page 259 of [2], that the starting point of the next solid body was the one he had described in the previous chapter (which, in this paper, is not Figure 2 but Figure 5). He also says that he divided its edges into three equal parts before removing the solid angles where the middle part terminates. Such mistakes may have occurred for lack of time or attention in revising the text but judging from the remaining description, Barbaro is describing the twice truncated icosahedron in Figure 3. He obtained this and the next solid body by truncating the vertices of two Archimedeans after dividing their edges into three equal parts, which yields new faces with twice as many edges as the initial. We obtain a regular hexagon if we connect the points from such a division on a triangular face. However, we get an irregular polygon if the initial face is regular and has four or more edges. This means the faces in the two solid bodies that Barbaro described could not be regular. Nonetheless, Barbaro stated it should have 60 triangles, 12 decagons, and 20 surfaces of 12 equal sides on page 90 of [1], so we divided the edges of the truncated icosahedron into unequal parts to obtain 20 regular dodecagonal faces in Figure 3(a). The remaining faces are irregular. Kaplan and Hart identify a similar case as a symmetrohedron on page 28 of [7]. As to the 3D print, to prevent the pipes from breaking during printing and make it more interesting, we extruded the triangular faces into short prisms and embedded them in the network of pipes.



(a) (b) (c) (d)

Figure 3: *Twice-Truncated Icosahedron in Chapter XX, page 99 of [1]: (a) spiegatura, (b) corrected planar net, (c) virtual model, (d) 3D print (diameter: 100 mm; pipe radius: 1.5 mm)*



(a) (b) (c) (d)

Figure 4: *Twice-Truncated Octahedron in Chapter XXII, page 101 of [1]: (a) spiegatura, (b) corrected planar net, (c) virtual model, (d) 3D print (diameter: 100 mm; pipe radius: 2.0 mm)*

The starting point for the twice-truncated octahedron shown in Figure 4 was the truncated octahedron. Although Barbaro does not mention any regularity, we chose its dodecagonal faces to be regular. Like the rectified truncated octahedron model, the pipes had to be slightly thicker to prevent them from breaking because of the wider faces.

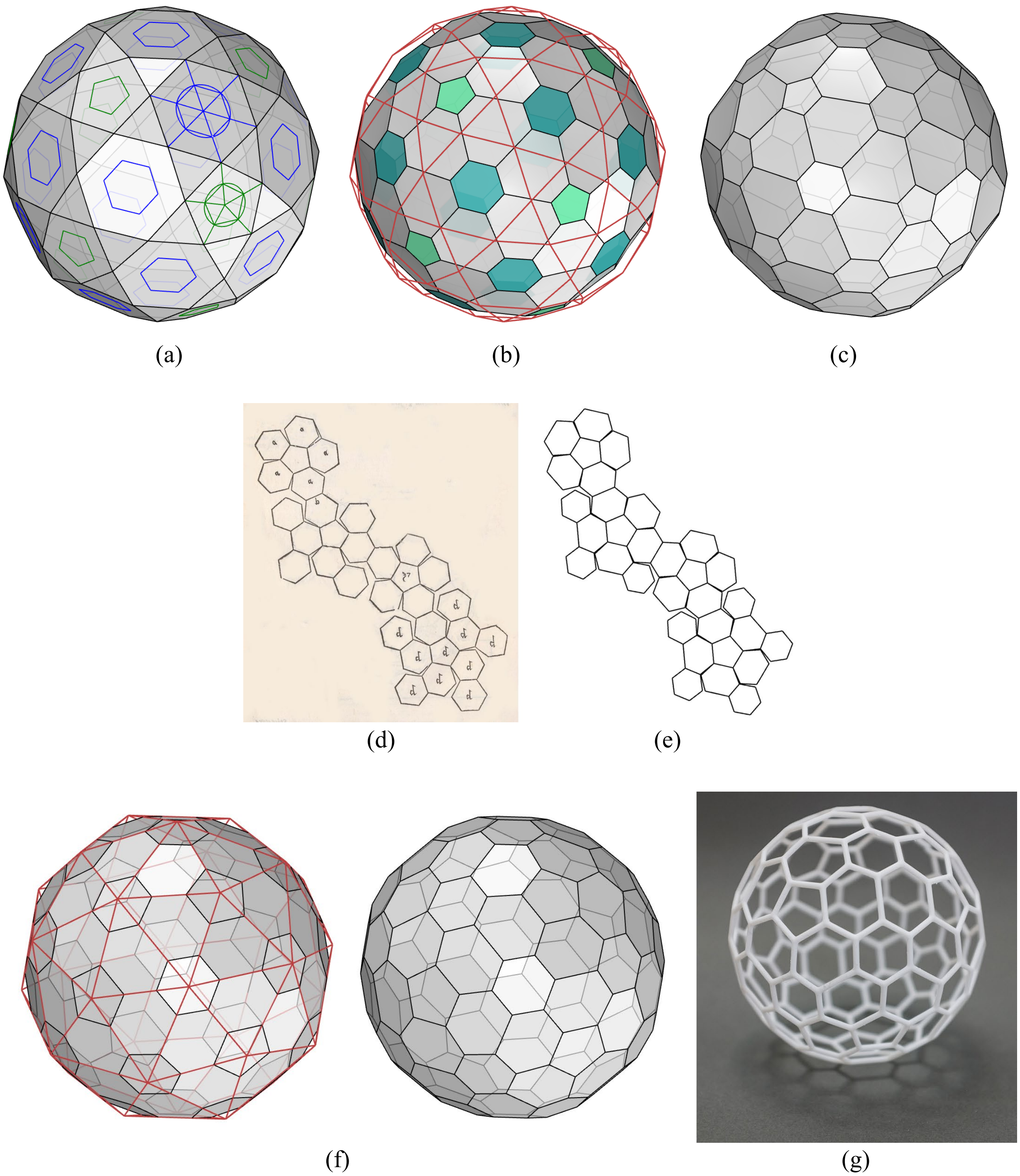
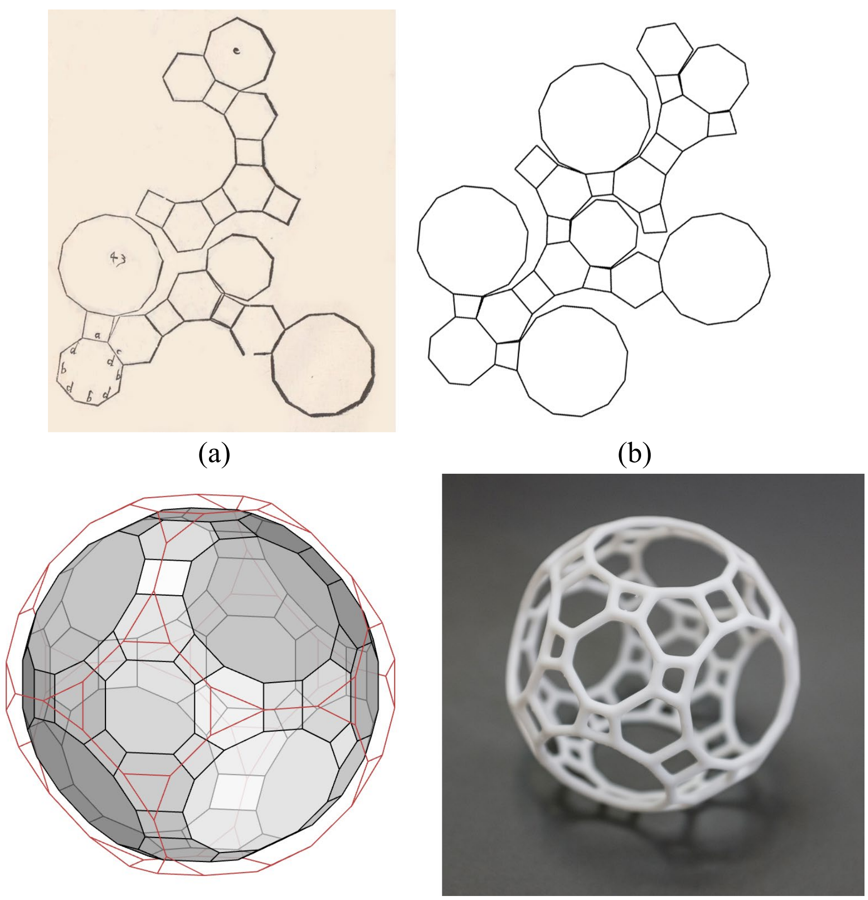


Figure 5: *Truncated Pentakis Dodecahedron in Chapter XIX, page 98 of [1]: (a-c) A possible procedure to replace the triangular faces of the rectified truncated icosahedron, (d) spiegatura, (e) corrected planar net, (f) virtual models, (g) 3D print (diameter: 100 mm; pipe radius: 1.0 mm)*

Barbaro does not explain how he obtained each of the following cases; he only says he replaced a set of faces with hexagonal ones: the triangular faces on the rectified truncated icosahedron and the triangular faces on the twice-truncated octahedron. For the first case, we show a possible procedure in Figures 5(a-c). We began by connecting the vertices of a pentagonal and a hexagonal face with its centroid. Joining the midpoints of each star segment gives us another pentagon and hexagon; replicating this in the remaining faces gives us smaller regular faces: 12 pentagonal and 20 hexagonal. If we suitably connect their vertices, we obtain hexagons, which are neither planar nor equilateral. Although it is unlikely that Barbaro ever envisioned such a procedure in a physical model, he could have drawn it in 2D from the *spiegatura* in Figure 2(a). Even if the outcome is not a polyhedron, it may have given Barbaro the idea that such a solution was possible. He may not have inferred that the hexagons could not be planar because their curvature, inwards, is very subtle. In his description, Barbaro does not distinguish the regular hexagons from the nonplanar ones, so it is possible that he understood them as if they were all regular. These hexagonal surfaces, however, "replace" the triangular faces of the rectified truncated icosahedron.

A similar solution with planar faces, shown in Figure 5(f), is obtainable by dividing the edges of the pentakis dodecahedron into three equal parts and subsequently truncating its vertices, as suggested by Pugh on pages 77-79 of [9]. The truncated pentakis dodecahedron Barbaro meant to describe is another near-miss Johnson Solid and Goldberg Polyhedron (3,0). Due to the length of the edges in this model, it was possible to 3D print the pipes significantly thinner than in other models with hexagonal faces.

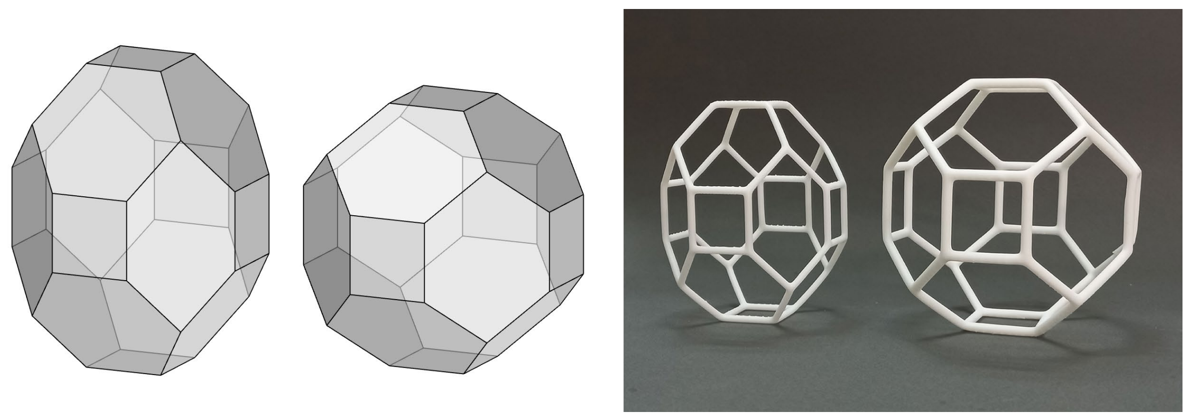


(a) (b) (c) (d)

Figure 6: *Truncated Rectified Truncated Octahedron in Chapter XXV, page 104 of [1]: (a) spiegatura, (b) corrected planar net, (c) virtual model, (d) 3D print (diameter: 100 mm; pipe radius: 1.75 mm)*

To interpret the *spiegatura* in Figure 6(a), we approached the problem parametrically [10], choosing the dodecagonal faces to be regular, and obtained the truncated rectified truncated octahedron in Figure 6(c). Instead of the square and rectangular faces that Barbaro predicted, there are rectangular and trapezoidal faces, respectively. Hexagonal faces have thus replaced the triangular faces of the rectified truncated octahedron. If we divide the edges of a truncated octahedron into three equal parts and truncate its vertices, we obtain a similar polyhedron with irregular faces. Since it has more edges than other polyhedra we have seen with large faces, the pipes did not have to be thick for the 3D print to work well.

On page 111 of [1], Barbaro says many other bodies are obtainable and mentions three examples without any drawings: one of six squares and twelve hexagons, another of 32 hexagons and squares six, and another of 90 squares, 60 hexagons, 12 ten angles, and 20 twelve angles. And if we continue this, he adds, the thing would end up taking us to infinity: "*Egli si potrebbe formare molti altri corpi simili, come sarebbe uno di sei quadrati, & dodici essagoni, & un altro di 32 essagoni, & quadrati sei, & un altro di 90 quadrati, 60 essagoni, 12 dieci anguli, & 20 dodicianguli ma la cosa andarebbe in infinito*". The remark is short but far from trivial and an additional testimony to Barbaro's creativity: similar to the cases in Figures 5 and 6, which went beyond rectification or truncation, Barbaro conceives something new. And what is all the more intriguing is that he did not even bother explaining how he had done it.



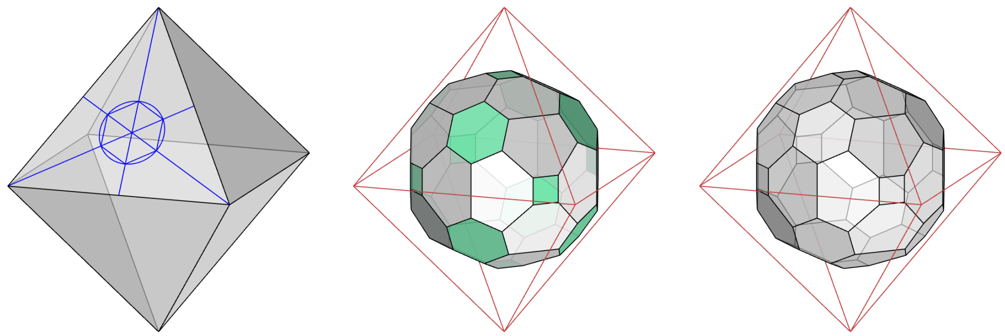
(a) (b) (c) (d)

Figure 7*: Interpretations of the first example in Chapter XXXIV, page 111 of [1]: (a-b) virtual models, (c) 3D print (diameter: 100 mm; pipe radius: 1.5 mm), (d) 3D print (diameter: 110 mm; pipe radius: 2.0 mm)*

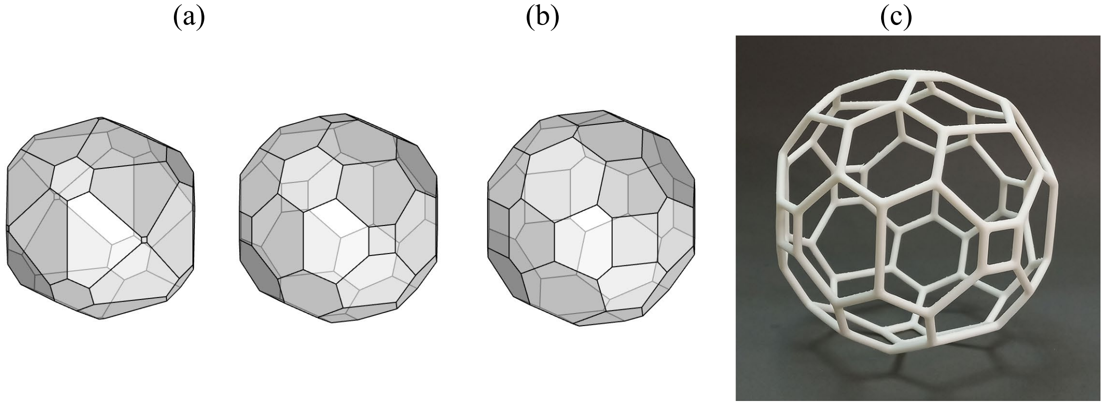
We interpreted the first example in [10] as an "elongated" version of the truncated octahedron that had been separated into two halves and connected with a ring of squares and hexagons, as shown in Figure 7(a). One of the reviewers of this article emphasized that such a solution was less symmetrical than usual for Barbaro and suggested he may have described a chamfered cube. Reflecting upon this remark was exciting because different versions of Barbaro's findings mean they are always open to discussion. Figure 7(b) shows another model based on the reviewer's remark. Both models in Figures 7(a) and 7(b) are equilateral and have 18 faces: 14 are regular on the left and, on the right, only 6, but the hexagonal faces are congruent. To model the chamfered cube, we uniformly expanded the faces of a cube, as shown in [11], concluding that the equilateral version is obtained when the distance between the cube's vertices and the intersection of three hexagonal faces equals half the cube's diagonal. In both 3D prints, the large, hexagonal faces raised numerous problems, which, once again, we solved with thicker pipes.

Figure 8 shows our interpretation of Barbaro's second example, which is yet to be named. In [10], we discussed a similar version with irregular hexagonal faces based on the model in Figure 7(a). Here, we consider the possibility that Barbaro may have used a procedure similar to the one in Figure 5(a-c) but with the regular octahedron as the starting point. Figure 8(a) shows the medians of the face in which we determined three vertices of a regular hexagon, halfway between the centroid and the closest edge midpoint. Figures 8(b-c) show that similar hexagons on the remaining faces give rise to six squares and 24 more irregular hexagons. By parametrically controlling the radius of the regular hexagons, multiple versions of this polyhedron are possible, some more interesting than others, as exemplified in Figure 8(d-f). Figure 8(g) is a 3D print of 8(e). As in previous cases, the faces of the octahedron were replaced by hexagonal faces, and the resulting polyhedron is strikingly different from any other in Barbaro's time.

The third example is a logical sequence to Figure 6, with the truncation of the vertices of the twice-truncated icosahedron. However, unlike Barbaro's prediction, there are not 90 squares but 30 rectangles and 60 isosceles trapezoids. The hexagonal faces were filled in the 3D print for a more interesting result.

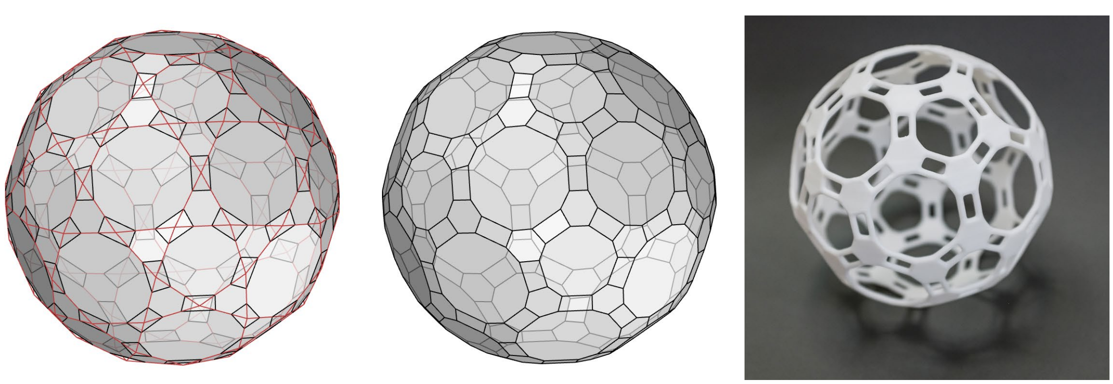


(a) (b) (c)



(d) (e) (f) (g)

Figure 8: *Interpretation of the second example in Chapter XXXIV, page 111 of [1]: (a-c) A possible procedure to replace the faces of the octahedron, (d-f) three different versions, (g) 3D print (diameter: 130 mm; pipe radius: 2.0 mm)*



(a) (b)

Figure 9: *Interpretation of the third example in Chapter XXXIV, page 111 of [1], Truncated Rectified Truncated Icosahedron: (a) virtual model, (b) 3D print (diameter: 130 mm; pipe radius: 1.75 mm)*

Summary and Conclusions

Before exporting the models as STL for printing, we modeled each polyhedron in *Rhinoceros* and *Grasshopper*. Considerable time was necessary for this because we had no prior experience in 3D printing, and we had to rework some models many times to avoid premature failures. Printing time could take 6 to 14 hours, depending on each model. After a few trials with entirely- or partially-filled prints, we chose to print the models without faces and replace the edges with cylinders to visually approximate the perspective drawings in *Pratica della Perspettiva*, whose visible and invisible edges were drawn with the same line type, as if there were no faces (in wireframe display mode). Printing these models as solid bodies or extruding their faces into short prisms would have been much easier, but it would not have been as challenging.

Barbaro pioneered as a visual mathematician, despite the inaccuracies found in *Pratica della Perspettiva.* Materializing with 3D printing the beautiful solid bodies he conceived in 1568 and holding them in our hands was exciting on many levels. We hope our research may inspire a renewed interest in Daniele Barbaro's work.

Acknowledgements

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