Three-Dimensional Diagonal Cross-Sections of Four-Dimensional Menger Sponges

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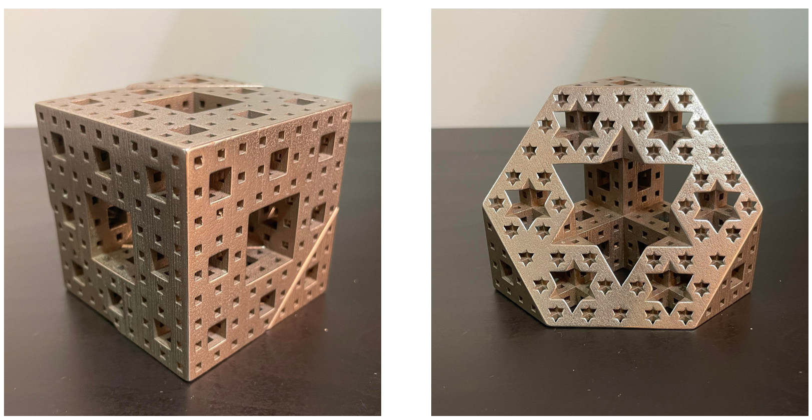
Abstract

When the 3D Menger sponge is sliced with a suitably chosen diagonal plane, a novel 2D fractal is obtained. In this work, I generalize this result by exploring the 3D fractal structures obtained by slicing two distinct 4D generalized Menger sponges with suitable hyperplanes. The resulting fractals are either etched in glass or 3D printed in precious metals and used to create fractal art. Analytical results are derived presenting the mathematics behind the art, including symmetries of the 3D cross-sections.

Introduction



(a) (b)

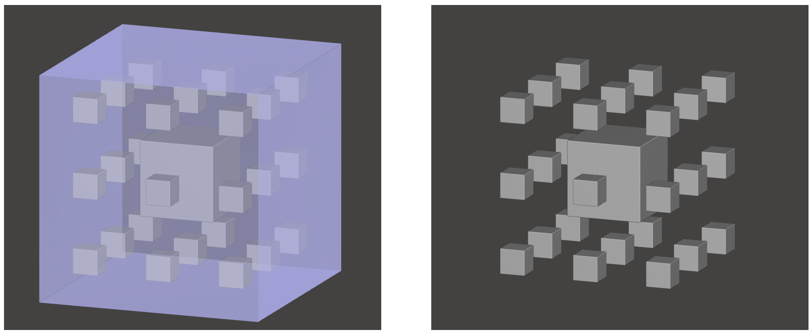
Figure 1: *3D Menger sponge fractal (with recursion depth three) 3D printed in steel in two halves held together with magnetism. In (a), we see the full sponge, while in (b) the two halves have been pulled apart. The diagonal cross-section reveals a novel 2D fractal.*

The Menger sponge is a well known fractal first described in 1926 by Karl Menger [8] (see [6, pg. 111-116] for an English translation), and may be viewed as a type of three-dimensional generalization of the Cantor set [7]. In 2007 it was shown by Sébastien Pérez-Duarte that a 3D Menger sponge sliced along an appropriately selected diagonal plane yields a novel 2D fractal consisting of a hexagon with a fractal pattern of "star of David" holes; see Figure 1 (or go to Sébastien's flickr page [2]) for an illustration. This discovery subsequently appeared in the New York Times [4], and an excellent explanation may be found in the video "Mathematical Impressions: The Surprising Menger Sponge Slice" by George Hart [1].

This begs the question - if a suitably defined 4D generalized Menger sponge is sliced with a suitably chosen hyperplane, might similarly interesting novel 3D fractals be found? It is the purpose of this paper to address this question, and to showcase some of the fractals that may be constructed in this fashion. A short film containing animations of said fractals can be found here [3].

Notation

- Π_c^n - the $(n-1)$-dimensional hyperplane given by $\{\vec{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = c\}$.
- C^n - the n-dimensional Cantor set (consisting of the 1D Cantor set Cartesian producted with itself n times). If $n = 1$ we write C rather than C^1.
- M_k^n - the n-dimensional Menger sponge of type k, defined below in Definition 2.



(a) (b)

Figure 2: *Computer rendering of the non-standard 3D Menger sponge M_2^3, with a recursion depth of two. The holes are all interior cavities invisible from the outside. In (a), we render the outer shell in transparent blue while the interior cavities are rendered in grey. In (b), only the interior cavities are shown.*

The n + 1 Possible n-Dimensional Menger Sponges and Some of Their Properties

The Menger sponge is typically defined by the following recursive procedure:

1. Start with a cube.
2. Divide the cube into a $3 \times 3 \times 3$ Rubik's cube of 27 smaller cubes.
3. Remove the middle cube from each face, as well as the cube in the center of the Rubik's cube.
4. Recurse on the 20 remaining cubes.

While intuitive and geometrically clear, this characterization does not yield efficient algorithms and is difficult to generalize to higher dimensions. Therefore, in this work I use an equivalent characterization based on the base 3 decimal expansions of the coordinates of points making up each fractal. To facilitate this, I make the following definition:

Definition 1. *Given $i \in \mathbb{N}$ and $x \in [0,1]$, we define*

$$\delta_i(x) = \begin{cases} 1 & \text{if the ith digit to the right of the decimal point in the base three expansion of } x \text{ is a } 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, the standard 3D Menger sponge M^3 may be defined as

$$M^3 = \{(x, y, z) \in [0,1]^3 : \text{for all } i \in \mathbb{N} \text{ we have } \delta_i(x) + \delta_i(y) + \delta_i(z) \leq 1\}.$$

With this in mind, for each $n \in \mathbb{N}$ I define $n + 1$ different possible n-dimensional Menger sponges:

Definition 2. *Let $n \in \mathbb{N}$ and let k be an integer obeying $0 \leq k \leq n$. Then we define the n-*dimensional Menger sponge of type *k as*

$$M_k^n = \left\{\vec{x} \in [0,1]^n : \textit{for all } i \in \mathbb{N} \textit{ we have } \sum_{j=1}^{n} \delta_i(x_j) \leq k\right\}. \tag{1}$$

Notice that for all $n \in \mathbb{N}$ we have $M_0^n = C^n$, the n-dimensional Cantor set, while $M_n^n = [0,1]^n$, the n-dimensional cube. The Sierpiński carpet and the standard Menger sponge are recovered as M_1^2 and M_1^3, respectively, while M_2^3 yields a non-standard 3D Menger sponge in which all of the holes are hidden cavities invisible from the outside - see Figure 2.

Intuitively, the parameter k controls the "hole-iness" of a Menger sponge. Considering the case $n = 3$, we note that on the one extreme, we obtain Cantor dust when $k = 0$, while on the other extreme, we obtain a solid cube when $k = 3$. When $k = 1$, we obtain the standard Menger sponge, which has enough holes to be interesting, but not so many as to be impossible to 3D print. Increasing to $k = 2$ results in Figure 2 - here there are plenty of holes, but they are all cavities invisible from the outside.

Remark 1. *It is worth mentioning that Karl Menger also considered n-dimensional Menger sponges of type k in his original work [8] and that his definition is equivalent to the one I have provided in Definition 2. Moreover, he goes further and proves that the integer k is the* topological dimension *(not to be confused with fractal dimension) of the resulting set. This fact does not seem to be widely known, however; I was unaware of it until late in this project.*

The following proposition generalizes the observation that each face of a standard 3D Menger sponge is a Sierpiński carpet, and that the holes of the 3D Menger sponge are given by three mutually orthogonal copies of the holes in the Sierpiński carpet cartesian producted with $[0,1]$ along a third dimension.

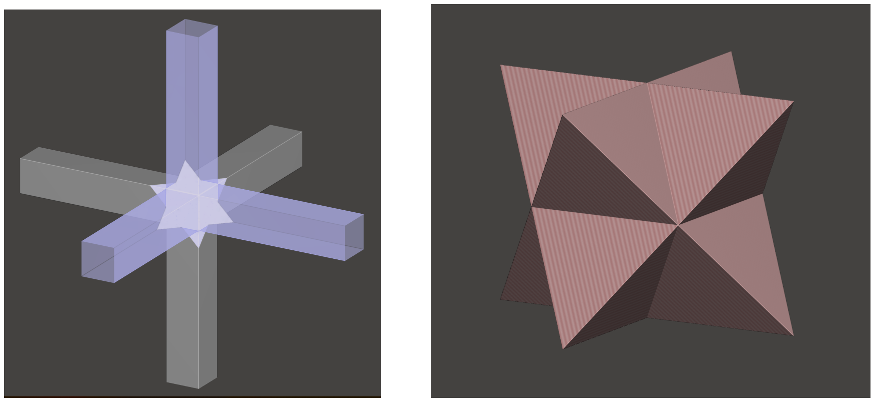
Proposition 1. *Suppose $k < n$. Then each face of a M_k^n Menger sponge is a M_k^{n-1} Menger sponge. Moreover, for $k < n-1$ the holes of an M_k^n Menger sponge are equal to the union of the holes in each face cartesian-producted with $[0,1]$ along the axis omitted from the face; hence they are visible from the outside. On the other hand, if $k = n-1$, the holes are interior cavities equal to $[0,1]\backslash C$ cartesian-producted with itself n times.*

Proof. By Definition 2, a point $\vec{x} \in M_k^n$ is part of a hole if $\delta_i(x_j) = 1$ for all $i \in \mathbb{N}$ for at least $k+1$ components of $\vec{x}$. So long as $k < n-1$, this leaves one free component which can be anything, from which the proof of the claim in the case $k < n-1$ easily follows. On the other hand, if $k = n-1$ there is no such free coordinate and therefore — as neither 0 nor 1 have have any 1s in their base 3 expansion — holes are not achievable on faces. It follows that every face is a copy of $[0,1]^{n-1} = M_{n-1}^{n-1}$, while any holes (if they exist) are interior cavities. Finally, the condition $\delta_i(x_j) = 1$ for all $i \in \mathbb{N}$ is equivalent to $x_j \in [0,1]\backslash C$, which shows that cavities *do* exist and are of the form claimed. □

From now on I recenter the Menger sponge at the origin and scale it by a factor of two, so that it is a subset of $[-1,1]^n$ rather than $[0,1]^n$.

Understanding the Diagonal Cross-Sections of the 3D Menger Sponge and its Generalizations: Configurations of n Mutually Orthogonal Cylinders in n-dimensional space

It is worth thinking about *why* we obtained the diagonal cross-section in Figure 1(b), and what we expect to obtain in higher dimensions. In a nutshell, Figure 1(b) is obtained because:



(a) (b)

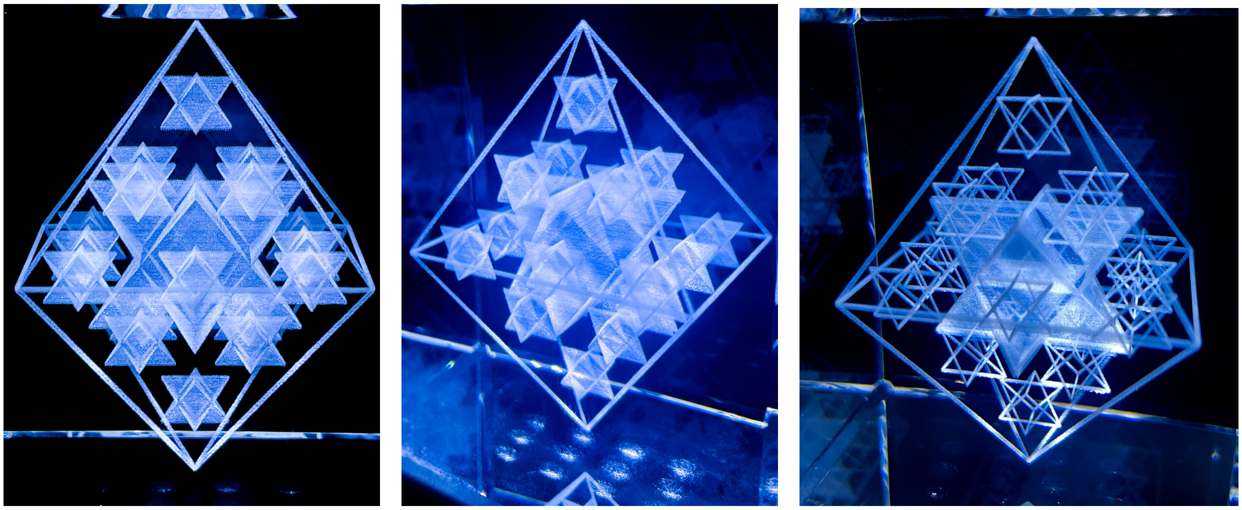
Figure 3: *In (a), we see that three mutually orthogonal square-based cylinders intersected with a suitable diagonal plane yields a hexagram. In (b), we show the result of four mutually orthogonal cube-based hypercylinders in $\mathbb{R}^4$ intersected with a suitable diagonal hyperplane — a stellated octahedron.*

1. A cube (axis-aligned and centered at the origin) sliced with the plane $x + y + z = 0$ yields a regular hexagon.
2. A collection of three mutually orthogonal square-based cylinders (axis-aligned and centered at the origin), when intersected with the same plane, yields a "star of David" (a compound of two triangles). See Figure 3(a) for an illustration.
3. The 3D Menger sponge is a subset of the cube and its holes contain many such triplets of mutually orthogonal cylinders, and many of them are centered on the plane $x + y + z = 0$.

When reasoning about 3D cross-sections of 4D Menger sponges, it is therefore reasonable to ask ourselves the following questions:

1. What 3D shape is obtained from a hypercube (axis-aligned and centered at the origin) when it is sliced with the hyperplane $x + y + z + w = 0$? Ans: an octahedron (I leave it to the reader to verify this and similar claims using their favorite mathematical software package).
2. What 3D shape is obtained by intersecting four mutually orthogonal cube-based cylinders (axis-aligned and centered at the origin) with the hyperplane $x + y + z + w = 0$? Ans: a stellated octahedron — also called stella octangula — a kind of 3D "star of David" consisting of a compound of two tetrahedrons with eight points in total [5, pg. 47-48]. See Figure 3(b) for an illustration.
3. Do the the holes in the 4D Menger sponge contain many such quadruplets of mutually orthogonal cylinders, and if so are many of them centered on the plane $x + y + z + w = 0$?

If the answer to the last question is "yes," then we might reasonably expect the 3D diagonal cross-section of a 4D Menger sponge to consist of an octahedron with a fractal pattern of stellated octahedron cavities. As we will we see shortly, this is true of exactly one of the five possible 4D Menger Sponges, and the resulting pattern is indeed what we expect. However, arguably even more interesting sections are obtained from a different variant of the 4D Menger sponge that *does not* meet the above requirement.



(a) (b) (c)

Figure 4: *3D cross-section of one type of 4D Menger sponge—namely M_2^4—with a diagonal hyperplane bisecting the hypercube, visualized as a glass engraving. The fractal consists of an octahedron (outer shell) with stellated octahedron interior cavities. There is one large star in the center, surrounded on all sides by smaller ones. For clarity, the outer octahedron is a wireframe, and in the rightmost photo the smaller stars are too.*

Symmetry of 3D Diagonal Cross-Sections of 4D Menger Sponges

Before examining the 3D cross-sections of two types of 4D Menger sponges in the next section, we first consider their symmetry. This will give us an idea of what to expect and help us to understand them.

Proposition 2. *The 3D cross-sections $M_k^4 \cap \Pi_c^4$ possess tetrahedral symmetry for all valid values of k. Moreover, if $c = 0$, octahedral symmetry is attained.*

Proof. Let us denote the cartesian coordinates of $\mathbb{R}^4$ by $\vec{x}_4$, and note that the plane Π_c^4 may be parameterized by 3D orthonormal coordinates $\vec{X}_3$ which are related to the coordinates $\vec{x}_4$ by

$$\vec{x}_4 = A\vec{X}_3 + \vec{b}, \quad \vec{X}_3 = A^T\vec{x}_4, \quad \text{where } A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \frac{c}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Any operation that leaves both M_k^4 and Π_c^4 invariant must also leave $M_k^4 \cap \Pi_c^4$ invariant. Clearly the group S_4 of permutations of the components of $\vec{x}_4$ does this for any $c \in \mathbb{R}$. To understand what this means for our 3D cross-sections, note that the permutation matrix $P_{ij} : e_i \longrightarrow e_j$ (here e_i is the ith basis vector of $\mathbb{R}^4$) induces the following map within Π_c^4:

$$\vec{X}_3' = A^T P_{ij} A \vec{X}_3.$$

Next, let $\vec{a}_i$ denote the ith row of A. It is an exercise in algebra to show that the above map takes $\vec{a}_i$ to $\vec{a}_j$ and vice-versa. Moreover, the set $\{\vec{a}_i\}_{i=1}^4$ forms the vertices of a tetrahedron, and hence the induced map above is precisely the set of symmetries of a tetrahedron. On the other hand, if $c = 0$ then the map $\vec{x}_4 \longrightarrow -\vec{x}_4$ also leaves $M_k^4 \cap \Pi_c^4$ invariant, and hence the induced symmetries within the 3D hyperplane are

$$\pm A^T P_{ij} A.$$

The set $\{\pm\vec{a}_i\}_{i=1}^4$ forms the vertices of a stellated octahedron, and the induced map above are its symmetries. A stellated octahedron has the same symmetry group as an octahedron, so for $c = 0$ we have octahedral symmetry.

□

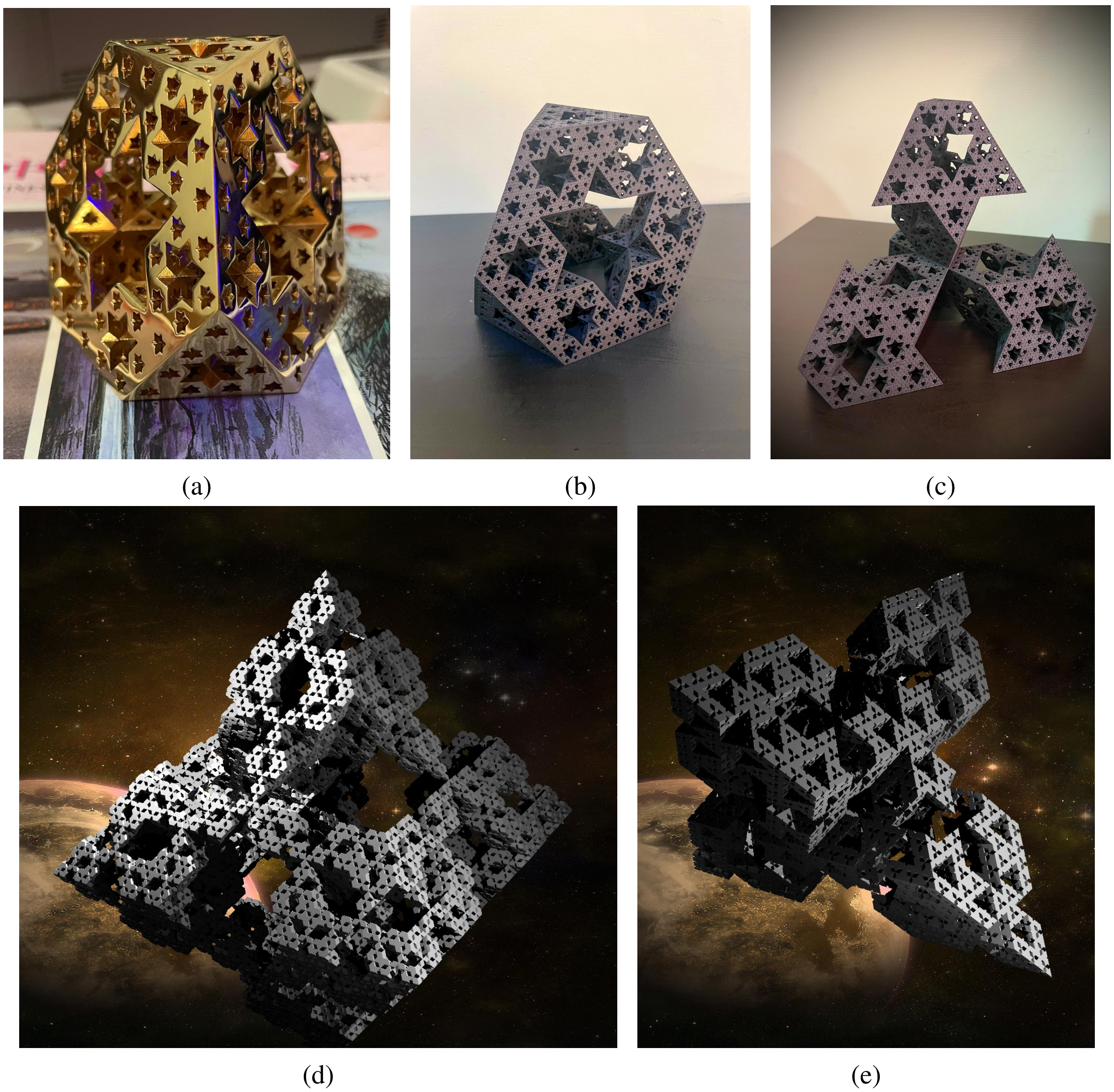


Figure 5: *3D cross-sections of a second type of 4D Menger sponge— namely M_1^4—with the hyperplane $x + y + z + w = c$. In (a) and (b) the $c = 1$ cross section is illustrated with 3D prints in two types of materials. In (c), we show a 3D print of the cross section with $c = \frac{5}{3}$, while (d) and (e) are 3D renderings of the sections with $c = 0$ and $c = -2.0659$ respectively. Notice that all sections have tetrahedral symmetry, while at $c = 0$ octahedral symmetry is attained. For animated versions of additional cross-sections, please see the short film [3].*

Two Varieties of 4D Menger Sponges and Their Respective 3D Cross-Sections

Specializing (1) to the case $n = 4$ and throwing out the cases $k = 0$ and $k = 4$ as uninteresting, we are left with three possible Menger sponges in four dimensions (corresponding to $k = 1, 2, 3$). The case $k = 3$ yields a simple fractal in which the central hypercube of a $3 \times 3 \times 3 \times 3$ hyper-Rubik's cube is recursively removed, and we similarly throw it away as uninteresting. This leaves us with the two choices M_1^4 and M_2^4 to work with.

We now ask ourselves "of these two possible 4D Menger sponges, which one meets the requirements of the discussion below Figure 3?" By Proposition 1, each face of M_1^4 is the standard 3D Menger sponge M_1^3, while each face of M_2^4 is the non-standard 3D Menger sponge M_2^3 shown in Figure 2. Also by Proposition 1, the holes in M_2^4 will consist of quadruplets of 4D cylinders each of which has a cube as a base - whereas the holes in M_1^4 will be more complex. It is therefore M_2^4—and not M_1^4—that we expect to give a fractal consisting of an octahedron with stellated octahedron cavities when intersected with the hyperplane $x + y + z + w = 0$. This is indeed the case, and the result is illustrated in Figure 4, by means of a glass engraving. Note the octahedral symmetry, as expected in light of Proposition 2.

This is arguably the most natural generalization of the 2D cross-section discovered by Sébastien Pérez-Duarte. Instead of a 2D six-sided filled convex polygon with a fractal pattern of six-pointed 2D star holes and hexagonal symmetry, we have a 3D solid eight-sided convex polygon with a fractal pattern of eight-pointed 3D star cavities and octagonal symmetry. However, being the most natural generalization isn't necessarily the same as being the most interesting generalization.

While the results for the 4D Menger sponge M_2^4 are intuitive and expected, those of M_1^4 are—to the author at least—surprising. Figure 5 gives examples of slices for a few values of c, some of them 3D printed, some of them 3D renderings. However, what I am able to illustrate in one figure is highly limited - see the short film [3] for additional cross-sections.

Notice that as expected in light of Proposition 2, all cross sections have tetrahedral symmetry. The value $c = 1$ gives a particularly nice result - the outer solid is a truncated tetrahedron with faces consisting of four regular hexagons and four equilateral triangles. The holes in this case are not interior cavities, but rather pierce the outer shell in such a way that each of the four hexagonal faces has—remarkably—the same pattern of holes as we found in the case of the diagonally chopped 3D Menger sponge M_1^3. In the next section, we explore the mathematics behind this strange result.

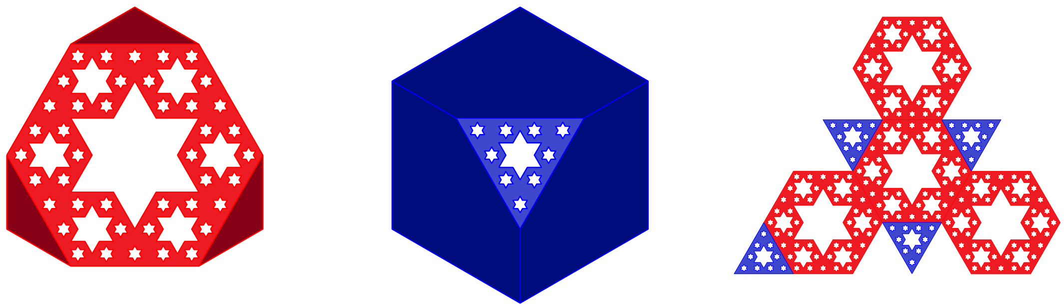
Relating Cross-Sections of n-Dimensional Menger Sponges to Those of (n − 1)-Dimensional Sponges

The key to understanding the cross-sections of M_1^4 illustrated in Figure 5 lies in the following theorem, in which an explicit relationship between diagonal cross-sections of M_k^n and those of M_k^{n-1} is derived.

Theorem 1. *The $(n-2)$-dimensional faces of of $M_k^n \cap \Pi_c^n$ consist of n copies of $M_k^{n-1} \cap \Pi_{c+1}^{n-1}$ and n copies of $M_k^{n-1} \cap \Pi_{c-1}^{n-1}$.*

Proof. The faces of $M_k^n \cap \Pi_c^n$ occur when $x_j = \pm 1$ for some $j \in \{1, \ldots, n\}$. Substituting $x_j = \pm 1$ into Π_c^n gives a copy of $\Pi_{c \mp 1}^{n-1}$, while $M_k^n\big|_{x_j = \pm 1}$ is a copy of M_k^{n-1} by Proposition 1. Hence, the n faces corresponding to $x_j = 1$ for some j are all copies of $M_k^{n-1} \cap \Pi_{c-1}^{n-1}$, while the n faces corresponding to $x_j = -1$ for some j are all copies of $M_k^{n-1} \cap \Pi_{c+1}^{n-1}$. □

Theorem 1 implies that $M_1^4 \cap \Pi_1^4$ has four faces that are copies of $M_1^3 \cap \Pi_0^3$ and another four faces that are copies of $M_1^3 \cap \Pi_2^3$. We already know that the former is a hexagon with a fractal pattern of star of David holes. The latter turns out to be an equilateral triangle (the sides of which are the same length as those of



(a) (b) (c)

Figure 6: *The 3D cross-section $M_1^4 \cap \Pi_1^4$ illustrated in Figure 5(a)-(b) has, by Theorem 1, four 2D faces equivalent to $M_1^3 \cap \Pi_0^3$ (a), and four 2D faces equivalent to $M_1^3 \cap \Pi_2^3$ (b). These may be joined together to form a 2D net (c) which, when folded together in 3D space, yields a truncated tetrahedron with a fractal pattern of star of David holes on every face.*

the hexagon) with a similar fractal pattern of star of David holes. These eight 2D shapes have to be woven together in 3D in such a way that, if we fill in the holes, the result would be a convex polyhedron with tetrahedral symmetry. The only possible way of doing this yields a regular truncated tetrahedron, with a fractal pattern of star of David holes on each face - see Figure 6. Indeed, this is exactly what we observed in Figure 5(a)-(b). However, it holds generically - similar reasoning applies to every cross-section in Figure 5.

Conclusions and Future Work

In this work I have generated fractal art based on the 3D diagonal cross-sections of two distinct varieties of 4D Menger sponge. Analytical results have been presented that help us to understand them. Theorem 1, which relates diagonal cross-sections of Menger sponges in 4D to those in 3D is particularly illuminating. Proposition 2, which examines the symmetry of the cross-sections, is also helpful. Work on an extension to the 5D case is already underway.

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