

Two-Disk Compound Symmetry Groups

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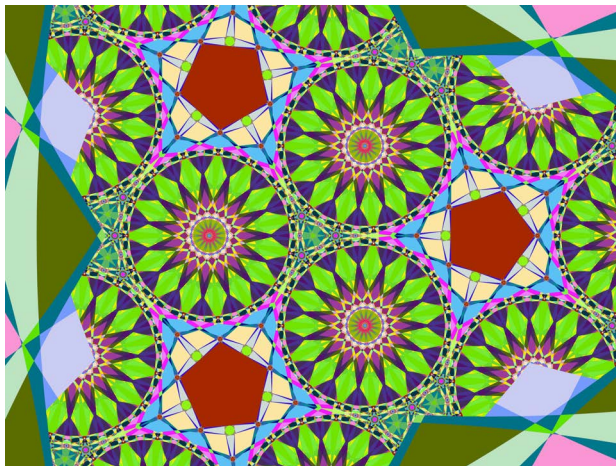
Abstract

Symmetry is at the heart of much of mathematics, physics, and art. Traditional geometric symmetry groups are defined in terms of isometries of the ambient space of a shape or pattern. If we slightly generalize this notion to allow the isometries to operate on overlapping but non-identical metric spaces, we obtain what we call *compound symmetry groups*. A natural example is that of the groups generated by discrete rotations of overlapping disks in the plane. Investigation of these groups reveals a new family of fractals, as well as a rich structure that is intriguing both mathematically and artistically. We report on our initial investigations.

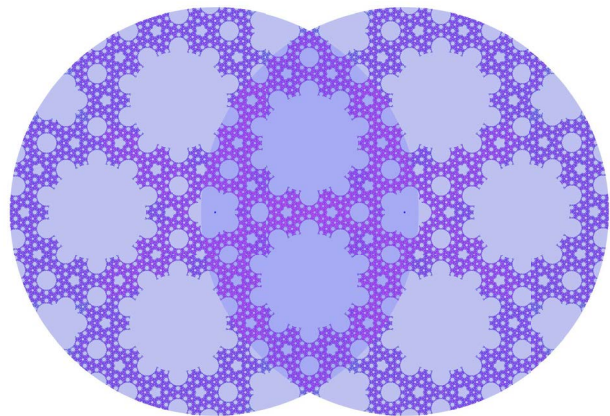
Introduction

Symmetry is of fundamental importance in many disciplines of mathematics, from the fields of Galois theory, to the automorphisms of abstract algebra, to the isometries of wallpaper patterns and crystals. In physics, Noether's theorem connects the symmetries of space and time with conservation laws. In art, subtleties of degrees and kinds of symmetry arguably lie at the heart of what constitutes beauty.

Here we expand on the traditional notion of geometric symmetry, with mathematical and artistic consequences, at least. The *symmetry group* of a shape or pattern is defined as the group of all isometries of the ambient space that preserve that shape or pattern. Isometries may only be combined in so many ways in any given metric space: there are (up to isomorphism) only 17 wallpaper groups, 7 frieze groups, 230 3-dimensional space groups, etc. We cannot have, for example, five-fold rotational symmetry in any repeating pattern in the plane—though quasicrystals can approximate this [9].



(a)



(b)

Figure 1: (a) Image of a compound symmetry group with 3-fold and 5-fold symmetries, (b) the $n = 5$ fractal.

By considering groups generated by isometries of metric subspaces that are not identical, but instead overlap, we can probe some of these “forbidden” symmetries in new ways. A partial image of one of these “compound symmetry groups” is shown in Figure 1(a). Locally, this image contains regions of three-fold and five-fold symmetry—necessarily broken on larger scales—as well as combinations such as 15-fold.¹ A new family of fractals is generated by these groups at critical parameter values, as shown in Figure 1(b).

In this paper we explore the characteristics of these new kinds of symmetry group. Of particular importance will be determining the parameter values at which the groups become infinite, exploring the underlying dynamics, and also understanding the characteristic fractals, or sometimes pseudofractals, that appear precisely at these transitions.



Figure 2: *The Gizmo Gears puzzle.*

This work began with the study of the behavior of certain “circle puzzles” [1, 2]—especially *Gizmo Gears* [7], shown in Figure 2, designed by Doug Engel. A two-disk compound symmetry group is the mathematical generalization of a circle puzzle. The study of the group structure of circle puzzles seems to have been initiated in [6]; related but simpler “wheel puzzles” are analyzed in [10].

Definitions and Basic Properties of Two-Disk Systems

A *compound symmetry group* is a group generated by a set of isometries of subspaces of a metric space.

Here we will primarily be concerned with compound symmetry groups generated by discrete rotations of two overlapping closed disks in the Euclidean plane. We sometimes call these *two-disk systems*. Without loss of generality, let the two disks be centered at $(-1, 0)$ and $(1, 0)$. Denote the left disk’s radius as r_1 , and the right disk’s as r_2 . The generators a, b are clockwise rotations of the left disk by $2\pi/n_1$, and of the right disk by $2\pi/n_2$. The group operation is function composition²: $fg(x) = g(f(x))$. We denote the group with these properties as $GG_{n_1, n_2}(r_1, r_2)$. If $n_1 = n_2$ we use a single subscript, and similarly for r_1 and r_2 . We can also omit the radius specification to indicate a family of groups with unspecified but equal radii. For example, one very important family is GG_5 —the groups generated by five-fold rotation of two equal disks. We will also be interested in (cyclic) subgroups generated by a single element, rather than by a and b . We indicate this with angle brackets: $GG_5(r)\langle ab^{-1} \rangle$ is the subgroup of $GG_5(r)$ generated by ab^{-1} .

A *portrait* of a compound symmetry group (or a single-generator subgroup) is a pattern that is invariant under the group elements. Most of our figures are portraits—Figure 1(a) is a portrait of $GG_{3,5}(2.42, 2.41)$.

To build some intuition about how two-disk systems work, consider Figure 3, which shows portraits of GG_5 at various r . Regions that remain connected under all elements (*pieces*) are colored identically; the color is a function of the size of the orbit. In Figure 3(a), $r < 1$, and the two rotations do not interact—the group is isomorphic to $C_5 \times C_5$. In Figure 3(b), the disks overlap, so there is some interaction. Viewed as a circle puzzle, we have added 9 pieces to the puzzle. This group is isomorphic to $C_5 \times C_5 \times A_9$ (we have added

¹Full portraits of all of the zoomed-in images in this paper are included in the online supplement, showing their context.

²We choose this convention so that move sequences can be read left to right. Similarly, the generators are defined to be clockwise rotations for compatibility with normal twisty puzzle notation.

the even permutations of the wedge pieces—see [10]). In Figure 3(c), the overlap has increased, and many more small pieces are created. Observe that in all cases, we do have five-fold rotational symmetry about two different points—but only within a fixed radius.

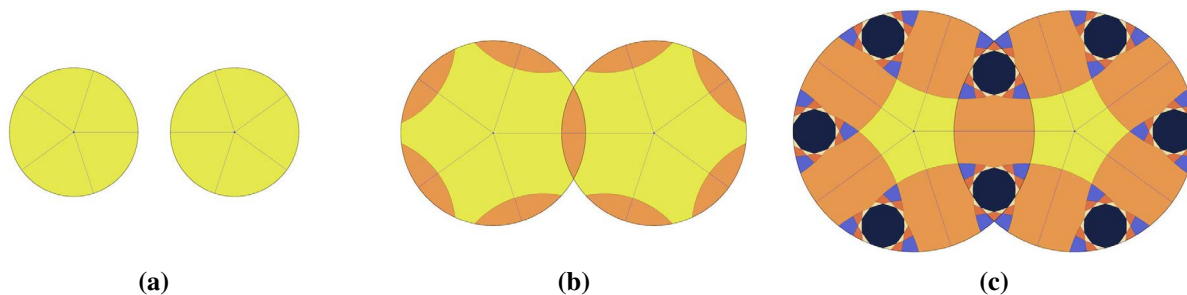


Figure 3: Different cases of GG_5 . (a) is isomorphic to $C_5 \times C_5$; (b) is isomorphic to $C_5 \times C_5 \times A_9$; (c) is more complicated.

While compound symmetry groups are generated by isometries, it is important to note that the general group element is *not* an isometry: If we perform ab , different regions have been rotated by different amounts about different centers. This is called a *piecewise isometry* [4].

Infinite Groups

A key question about any two-disk group is whether it is finite or infinite. If a family GG has some infinite member, it will have a *critical radius*, $r_c(GG)$, such that $GG(r)$ is finite when $r < r_c(GG)$, and infinite when $r > r_c(GG)$. (We believe, but have not proved, that it will also be infinite exactly at $r_c(GG)$.) We know this because the size of the orbit of any given point cannot decrease as r increases—all group elements that affect it are still available—so GG can never go from infinite to finite as r increases. We can also speak of the critical radius when $r_1 \neq r_2$ if we fix one radius.

We can precisely characterize which GG_{n_1, n_2} have infinite members (this characterization is closely related to the crystallographic restriction theorem):

Theorem 1. *There exists some r for which $GG_{n_1, n_2}(r)$ is infinite if and only if $\text{lcm}(n_1, n_2) \notin \{2, 3, 4, 6\}$.*

Proof sketch. We omit the “only if” proof in this paper, and prove the more interesting direction.

First, assume that $n_1 = n_2 = n$. Consider a point in the intersection of the two disks that remains in this intersection after being rotated counterclockwise about $(-1, 0)$ by $2\pi/n$. The group element $a^{-1}b$ will act as a simple translation on all such points. In particular, $a^{-1}b$ represents a translation along one side of a regular n -gon of circumradius 2, as shown in Figure 4 (consider, for example, the action on $(-1, 0)$); $a^{-2}b^2$ is another translation, and so on. We can generate translations from any one vertex of this n -gon to any other, as long as r allows the point to remain within both disks, by composing these sequences and their inverses.

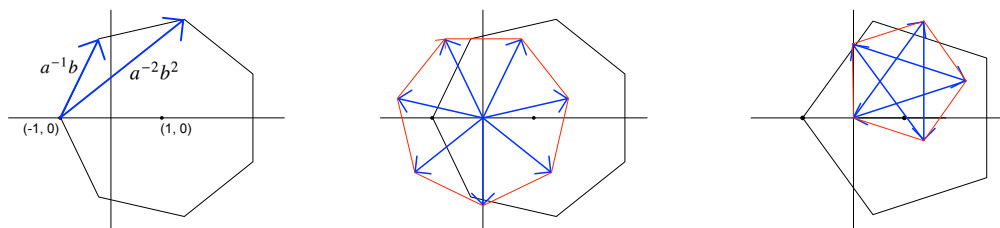


Figure 4: Constructible translations can shrink arbitrarily.

There are two cases. If $n > 6$, we take all the translations from one polygon vertex to an adjacent one, and observe the images we get of the origin under all of them. The resulting n points form another n -gon, but smaller than the original. But again, we can form translations between any of these new points by taking differences of the appropriate translations, and then repeat the process, resulting in a yet smaller polygon. Thus, we can generate arbitrarily small translations, and the group must be infinite. If $n = 5$, we start at the origin and apply every other pentagon edge translation, resulting in a pentagram shape whose vertices again form a pentagon smaller than the original. (See Figure 4, right panel.)

A careful inspection shows that no moves described above ever move a relevant point more than a distance of 4 from either disk center. Therefore, $GG_n(4)$ is infinite. If $n_1 \neq n_2$, then it is easy to show that $(a^{-1}b)^\alpha$ and $(ba^{-1})^\alpha$, for some α , are rotations by $2\pi/\text{lcm}(n_1, n_2)$ about two different centers. We can use these rotations in place of a and b in the proof above (with $r = 8$ instead of 4). \square

Geometric Constructions

For some n , we have geometric constructions showing that GG_n is infinite at a value of r matching our numerical estimates for critical radius. For other n we have plausible geometric constructions which agree well with our numerical estimates. But for most n , all we have is our numerical estimates.

The simplest case is $n = 5$. Figure 5(a) shows the relevant geometry: two regular pentagons are centered on $(-1, 0)$ and $(1, 0)$ such that the indicated edges lie along common lines, and the disk boundaries pass through the indicated vertices. These constraints yield $r = \sqrt{3 + \varphi} \approx 2.149$, where φ is the golden ratio. A similar analysis of the geometry in Figure 5(b), where $n = 10$, gives $r = \sqrt{4 - \varphi} \approx 1.543$.

Theorem 2. GG_5 is infinite at $r = \sqrt{3 + \varphi}$.

Proof sketch. We will show an explicit move sequence that maps the origin to an infinite sequence of distinct points. Referring again to Figure 5(a), interpreted now as the complex plane, let $\zeta_n = e^{2\pi i/n}$, and the point $E = \zeta_5 - \zeta_5^2$. Note that $|E + 1| = r$. We focus on how the line segment $E'E$ moves under specific sequences. The point $F = 1 - \zeta_5 + \zeta_5^2 - \zeta_5^3$ lies on $E'E$, as does the point $G = 2F - E$. We have three cases:

1. Line segment $E'F'$ is transformed by $a^{-2}b^{-1}a^{-1}b^{-1}$ to line segment GF .
2. Line segment $F'G'$ is transformed by $abab^2$ to line segment FE .
3. Line segment $G'E$ is transformed by $abab^{-1}a^{-1}b^{-1}$ to line segment $E'G$.

Together, these operations can translate any portion of the line segment $E'E$ piecewise onto itself. At no time does any point leave the intersection of the two disks during these transformations. The first two cases are translations of length $|F - F'|$, and the third case is a translation of length $|E - G|$. These two values are not rationally related to the total length $|E - E'|$, since $|E - E'|/|F - F'| = \varphi$. We can thus map the origin to successive points along $E'E$, by repeatedly choosing the transformation matching the region the point is in, indefinitely; it has an infinite orbit. (We omit due to space a proof that the single generator ab^{-1} produces the same behavior ($GG_5(\sqrt{3 + \varphi})\langle ab^{-1} \rangle$ is infinite), but note that this very interesting property—a single generator produces an infinite group at the presumed critical radius—is seemingly not shared by all n .) \square

For the cases of $n = 8$ (Figure 5(c)) and $n = 12$ (Figure 5(d)), their characteristic fractals (see the next section) can provide insight. A path of consecutive line segments that follows the fractal structure can be realized, starting from the center of one disk and approaching the disk boundary from the interior, each with a single turn. For $n = 8$, consecutive segments scale down by a factor of $\sqrt{2} - 1$. We can calculate the corresponding limit point from the path to yield $r = (5(2 - \sqrt{2}))^{1/2} \approx 1.711$. For $n = 12$, an analogous construction gives a scale factor of $2 - \sqrt{3}$, and $r = (2(20 - 11\sqrt{3}))^{1/2} \approx 1.377$. These conjectured closed-form radii rely on the limit point lying on the disk boundary at the critical radius, which has not been proven.

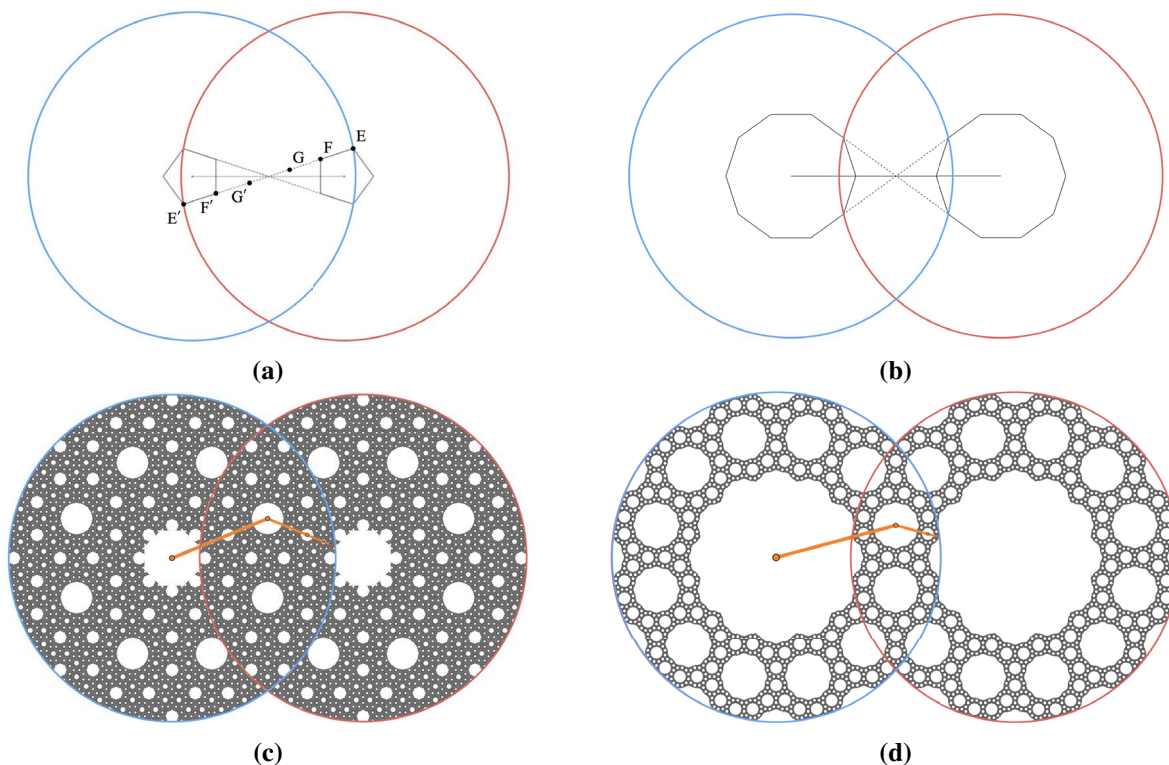


Figure 5: Geometric constructions for critical radius: (a) $n = 5$, (b) $n = 10$, (c) $n = 8$, (d) $n = 12$.

Critical Transitions and Fractals

Precisely at any group's critical radius, we observe a distinct fractal embedded in the image. It seems remarkable that these fractals have gone unnoticed for so long; they seem as natural as, for instance, the Mandelbrot set or the Sierpinski carpet. We can define the *characteristic fractal* for GG to be the set of points with infinite orbits at $GG(r_c(GG))$. In particular, this associates a unique fractal with every $n \notin \{2, 3, 4, 6\}$, and similarly when $n_1 \neq n_2$. We also have fractals when $r_1 \neq r_2$, but in that case the critical radius becomes a two-parameter family. The canonical example is the fractal for $n = 5$, shown in Figure 1(b). We also include in this paper the fractals for $n = 8$ (Figure 5(c)) and $n = 12$ (Figure 5(d)). In the online supplement, we include higher-resolution images of the characteristic fractals for all n up to 20.

The appearance of fractals here seems somewhat mysterious. The hallmark of a fractal is the repeating of a pattern on successively smaller scales. But unlike fractals constructed with an explicit recursive rule, there are no scaling operations in these systems, only rotations. Where do they come from? Why do they look so different for different n ? Some insight may be gained by considering Figure 4, which shows that in fact, rotations can combine to shrink patterns.

A natural question for any characteristic fractal is whether it is the closure of the orbit of a single point, or whether it consists of finitely or infinitely many disjoint closures of orbits. In many cases the fractals seem to be closures of the orbits of the origin, though there seems to be no theoretical reason this should be so. For $n = 23$, it appears that the fractal consists of two distinct pieces, symmetric about the x -axis. However, it is possible that these pieces are connected, and we have not been able to reach this connection numerically.

An especially interesting case is $n = 7$. Here, there seems to be a clear fractal structure. We can zoom in several times and see the same structure repeating. But then at a certain point, after zooming in by a factor of 500,000, the pattern suddenly changes. This was not apparent until we imaged the fractal to over 2 trillion

points. It is possible that this broken scale symmetry is a numerical artifact, but evidence argues against this. If it is real, this is perhaps the most mysterious process related to these systems we have yet observed. A video showing this transition may be found at [8]. A portrait of GG_7 just short of the critical radius, shown in Figure 6(a), reveals the characteristic $n = 7$ fractal motif.

In Figure 6(b), we see that embedded within GG_{12} lies a Koch-snowflake-like fractal seemingly based on four-fold symmetry, rather than the normal three-fold.

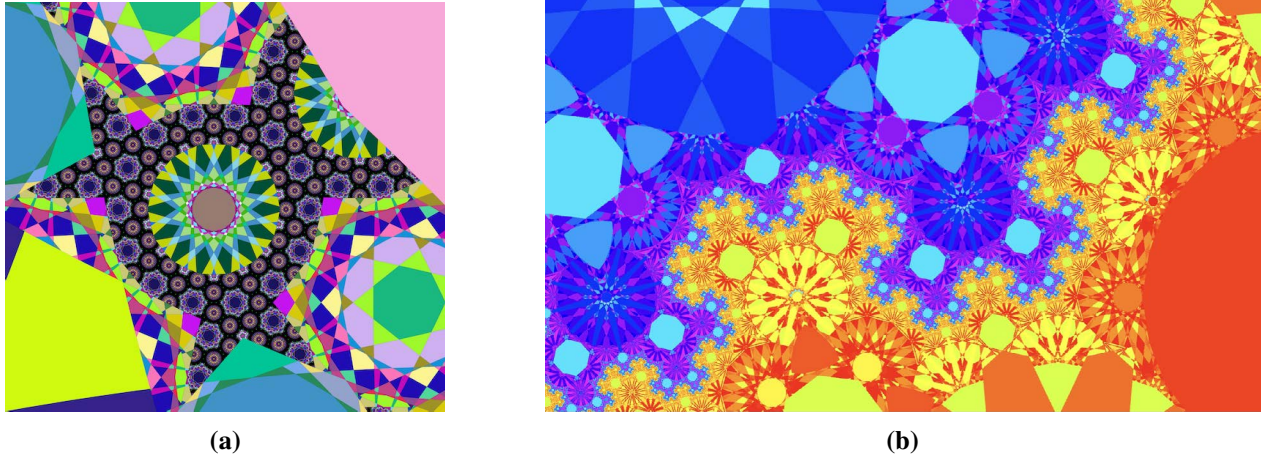


Figure 6: (a) A portrait of $GG_7(1.6233)$, (b) A portrait of $GG_{12}(\sqrt{2})(ab^{-1})$.

Numerical Models and Algorithms

All of the portraits in this paper were generated by programs which simulate compound symmetry groups. Conceptually, for each point we are interested in, we consider all possible rotation sequences applied to it, and plot the results—i.e., we plot its orbit. Figure 1(b), for example, was generated this way. When we want to show the action of the group on the entire space, it is more efficient to only process the disk boundaries. We process a boundary by discretizing it into small segments, processing those segments, drawing their images into a high-resolution bitmap, then filling the resulting spaces within the bounded regions with appropriate colors, according to some coloring rule. This is how Figures 1(a), 6(a), 7(a), and 7(b) were generated.

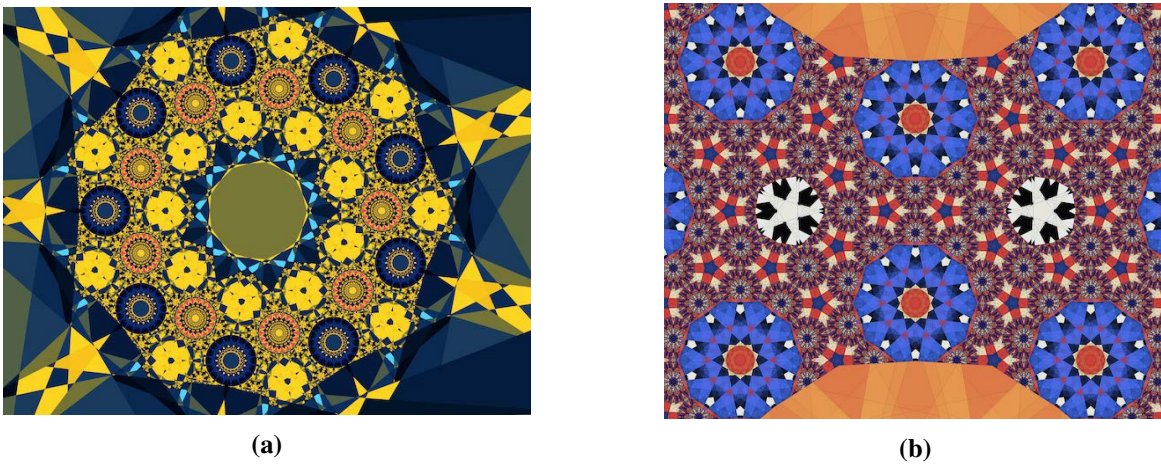


Figure 7: Filled images: (a) a portrait of $GG_9(1.408)$, (b) a portrait of $GG_5(2.144)$.

The use of frontier search [5] when generating orbits helps enormously, allowing us to not keep the entire orbit in memory. It has enabled orbits with more than 10 trillion distinct destinations to be computed, refining our critical radius estimates and exposing the curious behaviors of some fractals discussed above.

Single-Generator Images

Additional structure can be revealed by plotting orbits under a single generator, rather than all group elements. In the simplest case, we have $a^\alpha b^\beta$. This defines a discrete dynamical system which is an iterated piecewise isometry; Figure 8 shows examples. Some work has been done to characterize these systems, but many questions remain. [3, 11]

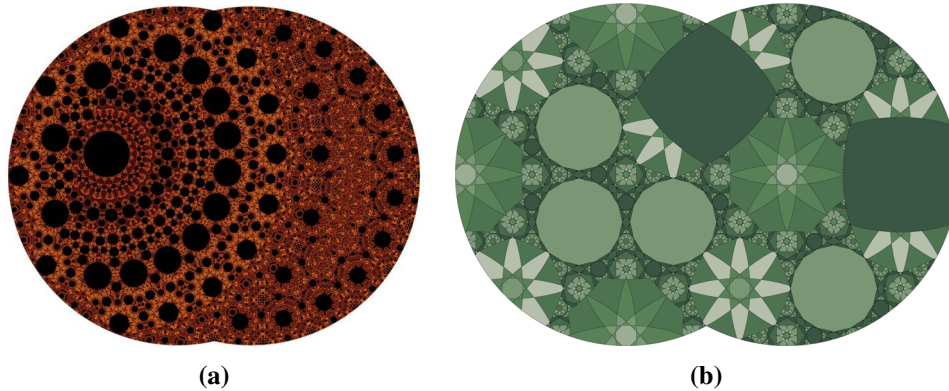


Figure 8: (a) The orbit of the upper intersection point is plotted for $GG_{21}(5.82)\langle a^2b^{-1} \rangle$ and colored according to its local density. (b) A portrait of $GG_8(2.12)\langle a^2b^5 \rangle$.

Critical Radii

Table 1 summarizes our knowledge of critical radii for GG_n with $n < 20$. A table up to $n = 100$ can be found in the supplement. In all cases, points were found with a minimum of 10 billion destinations, in some cases up to 10 trillion. These estimates may be too high because points with infinite orbit were missed, or too low because the points searched had very large but finite images. However, there is good agreement with the geometrically derived values, and we have confidence they are good to about 5 decimal places.

Table 1: Critical radii for GG_n .

n	Numerical estimate	Algebraic expression	Minimum Polynomial	n	Numerical estimate
5	2.148961	$\sqrt{3 + \varphi}$	$x^4 - 7x^2 + 11$	13	1.213594
7	1.623574			14	1.196554
8	1.711411	$\sqrt{5(2 - \sqrt{2})}$	$x^4 - 20x^2 + 50$	15	1.163276
9	1.408482			16	1.148470
10	1.543357	$\sqrt{4 - \varphi}$	$x^4 - 7x^2 + 11$	17	1.127509
11	1.290582			18	1.121505
12	1.376547	$\sqrt{2(20 - 11\sqrt{3})}$	$x^4 - 80x^2 + 148$	19	1.104246

Summary and Conclusions

The concept of compound symmetry groups opens up a new frontier in mathematics. Here we have just begun this exploration, by considering the two-disk compound symmetry groups. This investigation has revealed a new family of fractals, and a rich new source of spaces that combine symmetries in interesting and unexpected ways. Similar to “Seahorse Valley” in the Mandelbrot set, these are new “places” to explore, and we anticipate that there are more discoveries to be made. We must omit due to space many additional topics we would like to cover, such as the appearance of quasicrystals when we move beyond the critical radius, and other observations of aperiodic behavior.

While we have made significant progress in understanding two-disk compound symmetry groups, there is much more work to be done. In many cases, we lack even a basic theoretical understanding of the behaviors that cause the transition from finite to infinite size in these groups. For example, does every infinite two-disk group contain a point whose image is infinite, or a generator of infinite order? Is the critical radius of a two-disk system always an algebraic number, and is there a general formula for the critical radius? What dynamics are responsible for the creation of the fractals?

Similar questions can be raised more generally for multi-disk systems or arbitrary compound symmetry groups, where the behavior governing infinite size groups is more complicated. For example, we observe that Theorem 1 fails to generalize to three-disk systems: consider a set of three disks centered at, say, $(0, 0)$, $(1, 0)$, and $(\sqrt{2}, 0)$, and consider the compound symmetry group obtained by taking rotation increments $n_1 = n_2 = n_3 = 2$. Then if we choose disk radii to be sufficiently large, the resulting compound symmetry group will be infinite, because the three corresponding rotations of the plane generate a pair of translations along the x -axis whose ratio is irrational. Another question that presents itself is whether it is even decidable whether a given multi-disk compound symmetry group is finite.

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