Dancing with Dienes

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Abstract

Zoltán Pál Dienes (1916-2014), who spent the last years of his life living in Novia Scotia, the site of this year's Bridges Conference, was a mathematics educator and theorist and a pioneer in the use of manipulatives and embodied mathematics instruction. Dienes incorporated music, arts, and dance into math instruction materials, but more broadly constantly involved whole body movement and playful explorations in mathematics learning. I will examine some of Dienes' activities, see how Dienes included dances in them, suggest further possibilities, and attempt to focus renewed attention on his seminal work. I also document a 2009 dance performance that was inspired by a similar math class activity created by a noted follower of Dienes, Mary Laycock, and indicate how it also leads to Dienes-like class, studio, and performance activities.

Introduction

Zoltán Pál Dienes (1916-2014) was a mathematics educator and theorist and a pioneer in the use of manipulatives and embodied mathematics instruction. He spent the last years of his life living in Novia Scotia, about an hour from the site of this year's Bridges conference in Halifax. Dienes created numerous whole body, music, and dance class activities for students and published hundreds of papers and books on mathematics education and associated topics. Dienes often suggested that instead of using handheld mathematical manipulatives, teachers ask students to stand and move through patterns suggested by various mathematical structures. Dienes' substantial work on the use of embodiment strategies and connections to the arts in mathematics learning deserves to be more fully appreciated, but a complete review would take much more space than this paper allows. However, I will give an overview and will end by documenting a dance work and associated classroom and studio activities of the author's that was inspired by a Dienes-type workshop taught by an avid follower of Dienes' work, Mary Laycock, and which is briefly described in [14].

Zoltán Dienes was the son of Valeria Dienes, a prominent Hungarian dancer and choreographer who invented a somewhat mathematical dance theory and notation known as orchestics. At one point Valeria brought her family to live in the commune established in Nice and later Paris, France, by Isadora Duncan's brother Raymond, at which dance was an integral part of the schooling of young Zoltán. Zoltán's father Paul Dienes was a mathematician who was also a published poet and who had to flee for his life at the end of World War I due to his radical political activities. Interestingly, Dienes recounted that later in life he discovered that an influential childhood math teacher of his had himself been a student and protégé of his father Paul Dienes [8, pg 63]. Zoltán Dienes maintained a lifelong interest in artistic expression; he published a volume of poetry [6], articles on music and the arts [5,7], and the book *Mathematics through the senses, games, dance and art* [9].

Dienes' Movement Activities

Dienes often engaged students in unearthing, exploring, and playing with the mathematical structures hidden within the many games, stories, and activities with manipulatives and human movement that he created. But more than this, he was interested in students learning to recognize that similar structures exist in many disparate situations, and which he called the *multiple embodiment principle*. For example, in [10, III pp 15-21] he describes student activities for investigating the two groups of order six, the commutative cyclic group Z_6 and the non-commutative symmetric group S_3, the latter of which is isomorphic to the dihedral group D_3. In the following activities 1-5 Dienes employs a variety of physical and symbolic forms to model the cyclic group Z_6:

1. A circular clock face in which the hour hand is limited to multiples of two hour moves.
2. Tokens of three shapes, each one of two colors, and rules for moving them so one replaces another.
3. A word game using strings of the letters *A* and *B* and these rules:
   i. *AAA* can be removed from or placed within a string (i.e. *AAA* is the identity.)
   ii. *BB* can be removed from or placed within a string (*BB* is the identity.)
   iii. *AB* and *BA* can replace each other.
   These rules lead to six inequivalent possibilities: *A*, *B*, *AA*, *AB*, *AAB*, and the identity.
4. A word game using only *A* and leading to *A*, *AA*, *AAA*, *AAAA*, *AAAAA*, *AAAAAA* = identity.
5. Three equidistant points are marked on a circle and one student starts at a point facing into the center of the circle then moves one point clockwise but also turns to face away from the center. The next move is also one point clockwise but now the student turns to face into the center again. Six of these moves repeated alternately bring the student back to the starting point facing into the circle, and the six movements around the circle constitute the cyclic group Z_6.

Interestingly, as in items 3 and 4 above, Dienes constantly employs the use of letters and rules for manipulating them that often amount to solving group theory "word problems," the goal of which is often to decide whether two strings of such letters represent the same object. Yet in his writings he also rails against what he calls "symbol shock." In [16, pg 3] he declares that,

"One of the first things we should do in trying to teach a learner any mathematics is to think of different concrete situations with a common essence. (These situations) have just the properties of the mathematics chosen. Then … children will learn by acting on a situation. Introducing symbolic systems prematurely shocks the learner and impedes the learning of mathematics."

However, Dienes also usually finds ways to replace symbolic exercises by concrete activities. For example, in the section on the symmetric group S_3 in [10, III] he employs the following variations:

1. A movement game for three people standing at triangular vertices along with rules for how they may take each other's places in a way that creates the group S_3. For example, independent of the vertices where they currently are, persons *A* and *B* switch places, while person *C* does not move. An alternative set of rules refers not to specific people switching places, but people at specific vertices switching; for example, the people at vertices *A* and *B* switch, while the person at *C* does not move. Either formulation generates all six permutations and the full group S_3. Dienes is typically interested in helping students to notice that such distinct formulations of a problem lead to the same mathematical structure.
2. Working with a wooden or cardboard equilateral triangle with vertices marked *A*, *B*, and *C*, students manipulate the triangle and record which flips or rotations of the triangle leave which vertices in place.
3. As in the Z_6 item 5 activity above, students move between three points on a circle, facing either into or away from the center. Allowed moves consist of (1) rotating 180° in place, (2) moving one point to person's right – this may be clockwise or counterclockwise - and turning around, or (3) moving one point to the person's left and turning around.
4. A word game involving strings of letters *X* and *Y*, like the Z_6 word game in item 3 above:
   i. *XXX* can be removed, or else inserted anywhere.
   ii. *YY* can be removed, or else inserted anywhere.
   iii. *XYX* can be replaced by *Y* or *Y* can be replaced by *XYX*.
   Dienes has students play this word game using strings of counters colored either white or black.

In all the above Dienes employs his multiple embodiment principle, leading students to discover identical structures. He also pays attention to subgroups arising within Z_6 and S_3, and to the non-commutativity of S_3. Similar games to those above are also described for the Klein four group, Z_4, the order eight dihedral group of symmetries of the square, and even the order eight quaternion group.

Though the movements students create above involving symmetry groups are simply illustrations of mathematical structures, Dienes' writings often imply that in actual sessions with students they are encouraged to enjoy the dance aspects of the exercise. This is more clearly displayed in [9, Ch. 1] in which Dienes describes a circular trio in which dancers hold hands while they move through the six possible positions formed by S_3 transformations (Figure 1(a)). He describes a "twiddle" of dancer *C*, who remains in place as *B* passes under the outstretched arms of *A* and *C*, so that *A* and *B* exchange places. Since the dancers never drop hands, they will face outwards away from the center in three of the six positions and inward in the other three. Dienes suggests the dancers move through these positions using waltz steps (perhaps because they are familiar with such dance steps). He also proposes a variety of ways of creating dance movement with four such trios which permute their positions in four sections of the stage in accordance with the symmetries of the square, as they simultaneously play with the six triangular positions (Figure 1(b)). Such choreographic play with symmetry and the continuous holding of hands while dancing is common in a variety of folk-dance and even contemporary or classical dance forms. For example one might compare the uses of twelve dancers often in four trios or six duets in sections of Mark Morris' signature work "L'Allegro." In the same chapter Dienes describes similar choreography for four, six, or eight dancers, all maintaining handholds in circular formations as they explore possible configurations.

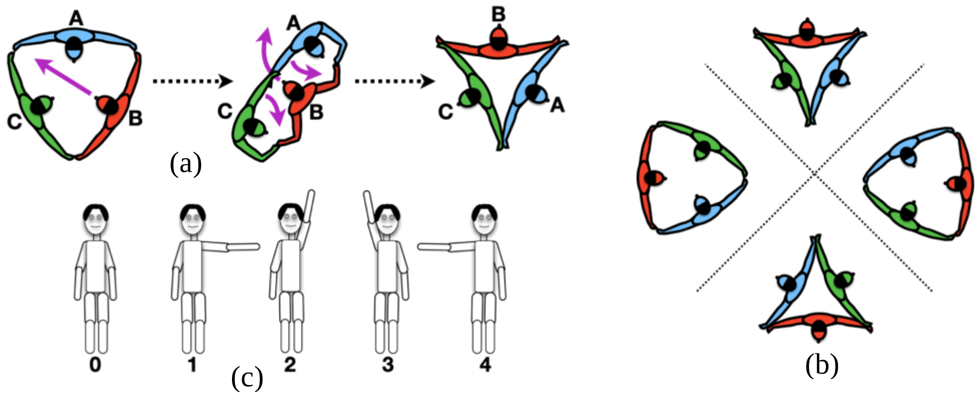


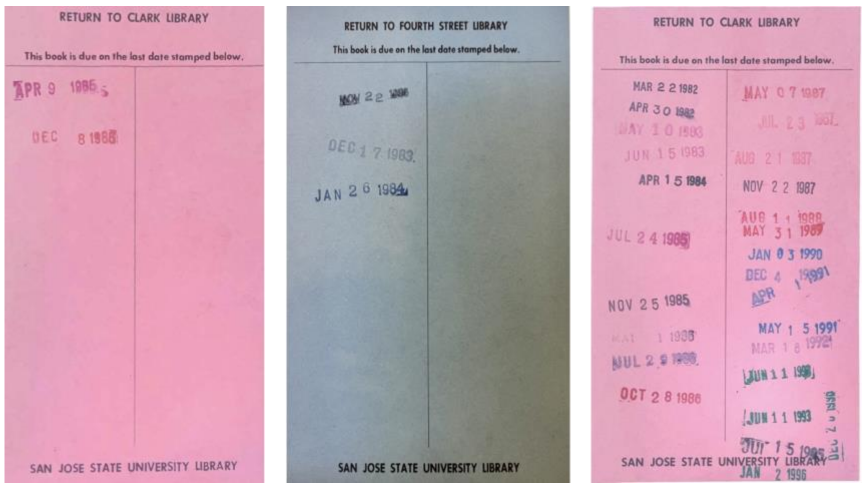
Figure 1: *(a) "Twiddle of C" has dancer B passing under arms of A and C as A and B trade places, ending with C remaining in place and dancers facing outwards. (b) Four trios allow for group explorations of square symmetries. (c) "Gymnastics with arms and legs."*

In chapter 2 of [9] Dienes invents several choreographic games which he calls "gymnastics with arms and legs." For example, Figure 1(c) shows five positions of arms in a mod five dance game. A "commander" signals dancers by displaying one of these positions and the dancers must all add that number, mod 5, to the position of their own arms. For example, if the commander stretches the right arm out to position 4, a dancer in position 3 must move to position 2, since $3 + 4 \equiv 2 \pmod 5$, though Dienes does not use modular arithmetic vocabulary. In the game format, participants start in a variety of positions, and anyone making a mistake must sit down. In the choreographic format Dienes suggests the commander gives periodic signals and the dancers take marching steps while executing changes displayed by the commander, who thus operates as a kind of movement conductor. The rest of this chapter explores a variety of additional and more complex arm and leg positions, cycles of lengths other than five, and multiplications as well as additions. Chapter 3 of [9] adds rhythmic patterning to the arm and leg gesture activities of chapter 2.

Dienes' Legacy

Are Dienes' works still widely appreciated? His multi-base arithmetic blocks, known as Dienes Blocks, are still widely used in elementary math education, and sold by many companies in a variety of colors and forms. If anything, the types of games, puzzles, manipulatives, and embodied activities he pioneered are found everywhere in classrooms, and embodied mathematics cognition is a hot topic among educational researchers. For many years I have checked out Dienes' books from the nearby San Jose State University

Library, and early on noticed that in the 1980s and 90s, when due dates were stamped on paper slips inside the covers, most of his books were rarely checked out. See Figure 2 for photos of checkout slips from three of his books for which the old slips have not yet been removed, which I checked out in January 2023. The use of these checkout slips seems to cover the years from the early 80s to the mid 90s. Two of the books had few checkouts, while *Let's Play Math*, written with Michael Holt had many. Dienes' writings are filled with mathematics games and activities, but his rules for engaging in them and his accompanying diagrams are often difficult to follow. This is not true of *Let's Play Math* [12], the co-author of which, Michael Holt, was an early and successful popularizer of math puzzles and games. The 74 games and activities in that book each have very short descriptive rules and little of the added analysis that Dienes often included in his writings.



(a) (b) (c)

Figure 2: *Three checkout labels of Diene's books from the San Jose State Univ, Library. (a) Concept Formation and Personality (1959), (b) An Experimental Study of Mathematics Learning (1963), (c) Let's Play Math (1973), with Michael Holt [12].*

In Dienes and Holt's section on Maze and Dance Games [12, pp 99-114], the authors state that psychologists have learned that "… children learn by moving about (they cannot, we now know, learn well when being made to sit still; if told to, they only fidget)" and "that child's play is in real earnest." Many of the games in [12] are quite similar in directions and philosophy to activities in Dienes' other books. For example, the Dance Games section includes triangle, square and non-square rectangle movement and dance games like those found with more analysis and development in [9].

In 2007 Bharath Sriraman published a volume of The Montana Mathematics Enthusiast devoted to documenting the impact and legacy of Dienes' work on mathematics education. The volume includes a copy of Dienes' unpublished 1995 book *Some Thoughts on the Dynamics of Learning Mathematics* [4], in which he summarizes and gives many examples of his work. Noting that the use of multiple modes of embodiment have become popular in mathematics education, Dienes states, "Such extensions are not only valuable as aids to the abstraction process, but also help the learner realize that the mathematics he or she is learning turns up in all sorts of unexpected places, and so helps give a reason for studying mathematics," [4, pg 9]. But he warns, "It is no easy task to construct interdisciplinary learning situations which do not place one or more of the disciplines in a subservient position" [4, pg 43]. In chapter IV on Interdisciplinarity, Dienes includes the first four columns of the table in Table 1, and mentions that, "Naturally, more "common threads" and more disciplines could be added to the above table…" — so I have added a column specific to dance to his categorization grid.

Permuting Dancers: "Switch"

Mary Chappell Laycock (1915-2011) [3] was a pioneer in the use of manipulatives and physical activities in math classes and was also a follower of Dienes. After spending many years teaching in Oak Ridge, Tennessee she moved to Hayward, California, and taught at the Nueva School in Hillsborough, California. She published hundreds of books and articles advocating for and demonstrating how to use hands-on activities and a discovery approach. For example, her book, *Straw Polyhedra* showed how to use straws and bobby pins to construct polyhedra very simply and inexpensively. A number of years ago I attended a workshop she taught at the California Math Council's Asilomar Conference, in which she had participants move through a permutation exercise that was very reminiscent of Dienes' activities. Standing three in a line, participants were required to cycle through all six ordered arrangements of three people in a line, allowing only two neighbors to switch places, never repeating a permutation, and ending back at the starting arrangement. This duplicates the structure found in "change ringing" of *n*-bell patterns in which only neighboring bells in a short sequence in which each of the *n* bells is rung once are allowed to switch places to play the next *n*-bell sequence.

Table 1: *First four columns are Dienes' "A Rough Measure of Extent of Disciplinarity"[4, pg 46]. I have added the DANCE column on the right.*

	MATHEMATICS	LANGUAGE	MUSIC	MOVEMENT	DANCE
TIME	Regular sequences, Generating functions, cycles	Phonetics, Verbal rhythm, rhymes, beats, subdivision of discourse	Musical rhythm, Bars, Musical phrases	Bodily rhythm, Walk, Skip, Jump	Dance phrases and sequences, sequencing through body parts, movement tempo, choreographic development, aspects of velocity/acceleration/jerk(rate of change of acceleration), impulse,
FORM	Structure, Symmetries, Isomorphisms, Transformations	Grammar, Syntax, Transformations, Poetic forms	Intervals, Scales, Harmony, Musical forms	Classify positions of dance, Gymnastic	Use of space and spatial qualities, spatial paths, floor patterns, group formations and symmetries, repeated patterns, classical dance forms, popular cultural dance styles, dance training patterns or forms, performance types, choreographic forms and tendencies, relation to performance space
CONTRAST	Inverses, Contradiction, Continuous vs. discreet, Formal vs. Heuristic	Drama, Rhythmical contrasts, Past vs. future	High-low, Piano-forte, Major-minor, Modulations	Up-down, Forward-backward, Fast-slow, Strong-weak, Left-right	Laban effort shape dichotomies: Flow (bound/free), Time (quick/sustained), Weight (strong/light), Space (direct/indirect), Shape (opening/closing, spreading/enclosing, rising/sinking, advancing/retreating, shape flow/directional/carving)
MEANING	Applications, Aesthetic aspect, Awareness of Structure	Semantics, Acting, Poetry reading, Invention of stories	Expression of message, Programmed music	Express sadness, gaiety, tragedy	Narrative vs. abstract, political, community (folk dance), exercise, history, athletic competition, psychological catharsis (dance therapy), religious or liturgical dance, social interaction (ballroom), sexual (erotic dance),musical form (tap), theatrical (musicals), ceremonial (whirling dervishes)
SYMBOLS	Symbol system of algebra, of geometry, computer language, invention of symbol systems	Reading and writing in the mother tongue and in other languages, Linguistics	Musical language (pentagram and other systems)	Choreography, Learning and inventing symbol systems	Choreographic symbol systerms:Labanotation, Benesh, Eshkol-Wachman, Beauchamp–Feuillet (baroque), Kahnotation (tap), computerized forms. Individual or personal notation forms. Verbal or written descriptions. Cue sheets. Visual or media arts representations (photos, painting, sculpture, video/film, animation)

The simple three-person exercise is easily accomplished, since only two moves are allowed from any position: either switching the first and second person, a "transposition" indicated by (12) using cycle notation, or else switching the second and third person, represented by (23). And since it is disallowed to repeat a sequence, one of the two possible switches would not be allowed (after the first switch).

After everyone in the workshop easily accomplished the three-person task, Mary asked us to do the same activity with four people in a row. In Figure 3(a) we show the "Cayley color graph" representing all possible switches between the 4! = 24 ordered arrangements of four dancers, here represented by the first letters of their names, *L*, *N*, *R*, and *K*, and with color coded edges representing either (12), (23), or (34). For simplicity the transpositions are abbreviated using *F* = First = (12), *M* = Middle = (23), and *L* = Last = (34). The dotted lines represent a "double switch" used in change ringing, combining *F* and *L*, represented as (12)(34) in cycle notation.

The four person puzzle is more difficult because at each step three switches are possible, (12), (23), or (34). After the first switch, one of those three must have been the preceding switch the use of which would repeat a position, so it cannot be used. Thus even though there are only two possible next steps, it is easy to find one's group at a dead end, unable to continue without repeating a position. The graph in which each of the 4! = 24 ordered arrangements is represented by a vertex and each transposition (12), (23), and (34) represents an edge is identical to the graph of the truncated octahedron and isomorphic to the graph known as the permutahedron — see [14] for diagrams and more details. A cycle that visits each vertex exactly once, such as that shown here by the solid arrows with beginning vertex *KLNR* is thus a Hamiltonian cycle for the graph and is called an "extent" by change ringers. For similarities between this type of problem, change ringing, contra dance, and the physics problem known as *n*-body choreographies, see [14]. Also see Tom Verhoeff's use of the same puzzle in [17].

In 2009 our dance company began work on a quartet that included the Hamiltonian cycle or "extent" shown in Figure 3 known as "Double Court Minimus" among change ringers and set it to a change ringing bell score by Leroy Clark. While looking for an overall meaning for the dance besides simply acting out a mathematical concept, one of the dancers who is an identical twin, and remarkably also has two identical twin brothers, began regaling us with humorous childhood stories about playing tricks on people using easy confusions involving identity switching. We included these stories in the piece and centered it around droll statements about DNA mutations and recombinant DNA sequencing, titling the piece "Switch." As

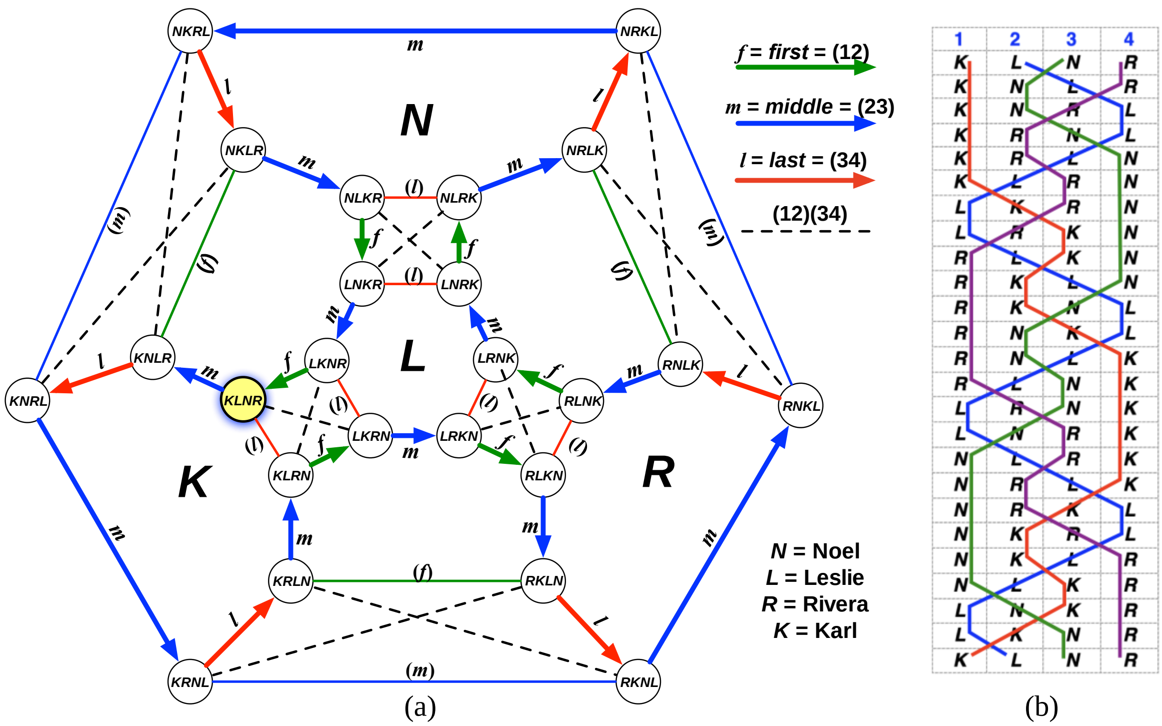


Figure 3: *Four dancers perform a Hamiltonian cycle through all 24 permutations using a change ringing "extent" known as Double Court Minimus, shown by bold arrows. Starting/ending vertex is KLNR.*

mentioned in [14], I opted to avoid any of the many change ringing extents that include the double switch (12)(34) to focus attention on each switch and had dancers remaining in place for any bell toll dance in a fluid manner as a metaphor for DNA strands floating in cells. However, in retrospect, I believe this created a very random looking piece, and in planning to recreate the dance I intend to use an extent that includes the double switch (12)(34) in which both pairs execute the same movements in order to create occasional symmetry and will replace the fluid movements by more dramatic action. Also, it is doubtful that more than a few audience members appreciated the presence of the change ringing problem, so I will add a theatrical element that more clearly calls attention to that aspect.

In this extent, one of the dancers, Leslie, executes a pattern known as "plain hunting" while each of the other three follow an identical pattern in canon that is different from Leslie's, see Figure 3(b). Every other switch in this extent is (23) or *m*, which simplifies the memorization task for the performers. Change ringers would start this extent at vertex *LKNR* with plain hunter Leslie in the first place, rather than where we started it at *KLNR*. Notice that each of the hexagons labeled *K*, *N*, and *R* have that person remaining in place 1 while the other dancers execute the six-vertex extent for the other three; *L*'s hexagon is returned to periodically, two vertices at a time. An algebraic notation for this extent is $(ml)^2(mf)^2$, which is understood to repeat three times. See [15] for a video excerpt of the dance section showing the 24 transpositions. The text and a more detailed description of the choreography is in the supplement to this paper.

The permutation problem for three or four people makes for an entertaining and enlightening class activity for Discrete Math or Liberal Arts math classes which include graph theory or polyhedra topics (see the handout in the supplement to this paper). Some teachers might prefer not displaying the planar "Schlegel diagram" of the graph shown in the handout, asking students to discover it on their own. I instead want the students to spend more time working on how they will perform the Hamiltonian cycle. One of the most entertaining quartets in one of my Discrete Math classes was performed by a trio who had lost their fourth member, and who grabbed an umbrella and hilariously executed all 24 permutations by three people and the umbrella. A warmup activity I sometimes use asks students or workshop participants to perform the two partial paths in a Hamiltonian cycle in one of the 4-person change ringing graphs in [14], using simple gestures easily performed while seated, shown in Figure 4(a) and (b). "Clap Trap Too" alternates with double switches shown by the blue arrows. Sometimes we duplicate the change ringing rules by dividing the class into four groups, one of which performs all the taps, another all the claps, etc. Or try variations on the Figure 4(a) and (b) patterns. If brave, try to perform an entire Hamiltonian cycle of the 24 sequences.

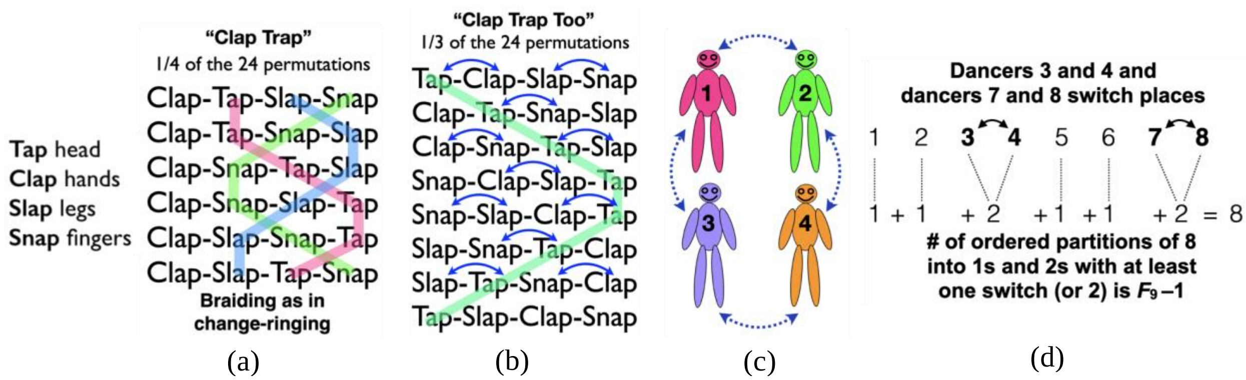


Figure 4: *(a), (b) Sections of Hamiltonian change ringing cycles that can be performed by seated participants. (c) Four dancers in a square may involve many more transformations. (d) An example for 8 dancers illustrates why the n-dancer graph has degree F_{n+1} –1 [13].*

A variety of additional class, workshop, or performance activities may be found by perusing Dienes' many writings for mathematical movement ideas. Even those that seem only to involve symbols are easily transformed to dance or movement in the same manner that Dienes used: let each symbol be represented by a person and replace operations on the symbols by movements. For example, arrange four people in a square as in Figure 4(c), allowing switches (12), (23), (34) but also (41). This is very close to several of Dienes' activities, such as in [9] and [12], but he also allows other switches such as (12)(34) and (13) which are common in square dance and other forms. Students may be surprised to see a connection to the Fibonacci sequence arising out of the change-ringing switch patterns: figure 4(d) illustrates how the degree of vertices in the n-person change ringing graph counts the number of ordered sums of 1s and 2s yielding n (not including all 1s). Here the permutation (34)(78) corresponds to 1+1+2+1+1+2=8. See [13] for detail on why this generates the n+1st Fibonacci number minus 1 and a discussion of connections between such ordered sums of 1s and 2s and dance, poetry, and rhythm patterns that arose in ancient India.

Others have developed similar materials. Frequent Bridges contributor Tom Verhoef and choreographer Roos van Berkel have explored several combinatorial dance and movement projects like those above [2, 17]. Verhoeff uses color coding of performers and solves associated mathematical and movement problems. Professional folk, Flamenco, and ballet companies perform several computer algorithms through dance at [1] using sequences of two-dancer switches, and these might interest students. Jim Henle has explored ways of combining dance and mathematics in his Mathematical Intelligencer column, especially with reference to Dienes' work such as in [11], and reports on his and his students' similar projects; these might inspire other classroom or studio experiments.

Conclusion

Throughout his pioneering career in the field of mathematics education, Zoltan Dienes incorporated dance and movement activities in his work. Emphasizing the playful and engaging aspects of dance over artistic creation, he pioneered embodiment as a bridge between the concrete and the abstract. The numerous games, movement puzzles, and manipulative activities that he developed are deserving of renewed attention. In addition to being a wonderful source of ideas for contemporary math educators, his work may also inspire the creation of new work blending mathematics with the movement arts.

References

[1] AlgoRhythmics. *Algorithms*. https://algorythmics.ms.sapientia.ro.

[2] R. van Berkel and T. Verhoeff. "Lehmer's Dance – a Lecture Performance." *Bridges Conference Proceedings*, Linz, Austria, Jul. 16-20, 2019, pp 375-378. https://archive.bridgesmathart.org/2019/bridges2019-375.html#gsc.tab=0.

[3] California Math Council. *Mary Laycock*. 2015. https://www.cmc-math.org/mary-laycock

[4] Z. P. Dienes. *Some Thoughts on the Dynamics of Learning Mathematics*. The Montana Mathematics Enthusiast: Monograph 2, edited by B. Sriramen, 2007, pp 1-118. https://www.zoltandienes.com/wp-content/uploads/2018/08/ZPD_and_Dynamics_of_Math_Learning-Monograph2_2007.pdf

[5] Z. P. Dienes and C. Longuet-Higgins. "Can musical transformations be implicitly learned?" Cognitive Science, 28 (2004) pp 531-558.

[6] Z. P. Dienes. *Calls from the Past*. Upfront, 2003.

[7] Z. P. Dienes. *Mathematics as an art form*. 2002, 2004. https://zoltandienes.com/wp-content/uploads/2010/05/Mathematics_as_an_art_form.pdf.

[8] Z. P. Dienes. *Memoirs of a Maverick Mathematician*. Minerva Press, 1999.

[9] Z. P. Dienes. *Mathematics through the senses, games, dance and art*. NFER Publishing Company, LTD, 1973.

[10] Z. P. Dienes and F. W. Golding. *Geometry through transformations. I. Geometry of Distortion*. *II. Geometry of Congruence*. *III. Groups and Coordinates*. Herder and Herder, 1967.

[11] J. Henle."Mathematics that Dances." The Mathematical Intelligencer, 43(4), pp 73-79, 2021.

[12] M. Holt and Z. P. Dienes. *Let's Play Math*. Walker and Company, 1973.

[13] K. Schaffer. "Dancing with Fibonacci." *Proceedings of the 12th International Conference on Mathematical Creativity and Giftedness*. Aug. 25-28, 2022, pp 314-320. https://d-nb.info/1268386723/34.

[14] K. Schaffer. "Dances of Heavenly Bodies: Dance, *N*-Body Choreographies, and Change Ringing." *Bridges Conference Proceedings*, Enschede, Netherlands, Jul. 27-31, 2013, pp 543-546. https://archive.bridgesmathart.org/2013/bridges2013-543.pdf.

[15] K. Schaffer. "Switch." 2009. Dance. Santa Cruz. https://vimeo.com/795156278/c310517c16.

[16] B. Sriraman and R. Lesh. "Reflections of Zoltan P. Dienes on Mathematics Education." The Montana Mathematics Enthusiast: Monograph 2, edited by B. Sriramen, 2007. For URL see [4].

[17] T. Verhoeff. "Combinatorial Choreography." *Bridges Conference Proceedings*, Towson, Maryland, Jul. 25-28, 2012, pp. 607-612. https://archive.bridgesmathart.org/2012/bridges2012-607.html#gsc.tab=0.