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Curved, yet Straight: Stick Hyperboloids

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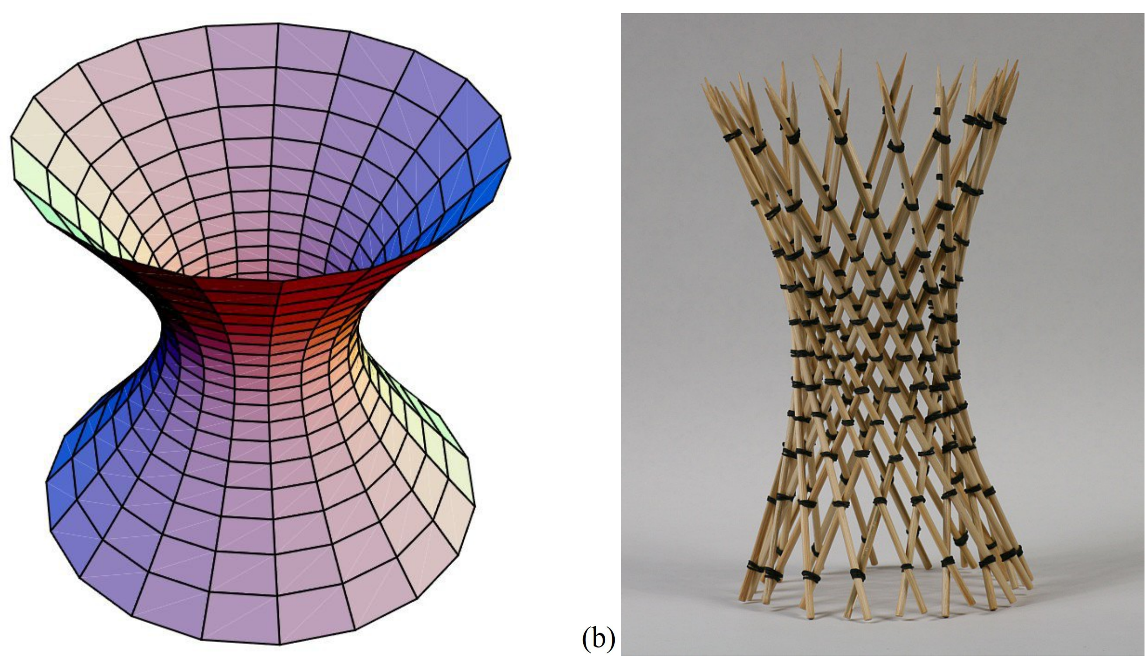
Abstract

This paper discusses "stick hyperboloids" and three geometric questions related to their construction. These are physical structures made by assembling rigid physical rods that cross each other, creating an attractive model of a hyperboloid of one sheet. First, some educational, architectural, and artistic examples are illustrated. Then original results are explained concerning the spacing between the crossings, the angles for pre-drilling connector holes, and what exactly happens when the structure is flexed. These results almost certainly were discovered previously, but I could find no hint of them in the literature. In addition, I show a new form of paper hyperboloid model that emerged from this analysis.

Introduction and Examples

The hyperboloid of one sheet is a classic surface in 3D space. [16] I will ignore the hyperboloid of two sheets in this paper, so from here on "of one sheet" will always be implied. Figure 1(a) illustrates a finite portion of this infinite hourglass. It is a quadric surface, so can be expressed as a polynomial equation in the variables x, y, and z using exponents no higher than 2. A standard form is $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$, where the parameters a, b, and c determine its shape. A common special case is when $a = b$, making it rotationally symmetric about the z axis, i.e., a surface of revolution with circular (instead of the generally elliptical) "waist." It can be generated by rotating a hyperbola about one of its mirror lines. Hyperboloids are "doubly ruled" surfaces, meaning through each point of the surface it is possible to draw two distinct straight lines entirely within the surface. One set of rulings is right-handed while the other is left-handed. "Stick hyperboloids" are physical models, as in Figure 1(b), that use rigid rods to reify these rulings. They are visually engaging in part because they manifest the apparent paradox of being doubly curved surfaces made of straight components.



(a) (b)

Figure 1: *(a) finite portion of a hyperboloid surface. (b) stick hyperboloid of skewers and rubber bands.*

The history of hyperboloids goes back to classical times. After calculating the volume of a sphere, Archimedes described how to calculate the volume of a segment of a hyperboloid. [13] The polymath architect Christopher Wren discovered and formally proved in 1669 that the hyperboloid is a doubly ruled surface. [1] (As far as I know, Wren did not use hyperboloids in any architectural context.) He described how the properties of ruled surfaces simplify the grinding of hyperboloid-shaped lenses, as one can apply a straight tool obliquely to a rotating cylinder of glass. Wren was inspired by seeing a round wicker basket “woven only from straight pieces of osier lying at oblique angles”, presumably akin to Figure 1(b) and the modern wicker-ware mentioned below.

The geometry and algebra of quadric surfaces, including ellipsoids, paraboloids, hyperboloids, etc., has standardly been taught as part of the 3D analytic geometry and multivariate calculus curriculum. They provide an excellent domain for connecting formal methods with visual understanding. As aids in such mathematical pedagogy, plaster casts of hyperboloid surfaces and string models of the rulings became common in late 19^{th} century and early 20^{th} century university education. Many historical collections and educational catalogs include them. (Paper “sliceform” models of hyperboloids were also made, but are not discussed here.) Figure 2(a) is a 19^{th} century plaster model of a hyperboloid surface (with elliptical cross section) from the mathematical models collection of the National Museum of American History. Figure 2(b) is a string model from the same collection showing the rulings of a hyperboloid, while also including a concentric string model of the asymptotic double-cone within it. [15] With modern technology, I have made 3D-printed stick hyperboloids as educational models. Figure 2(c) shows a two-color example anyone can download and replicate. [11]

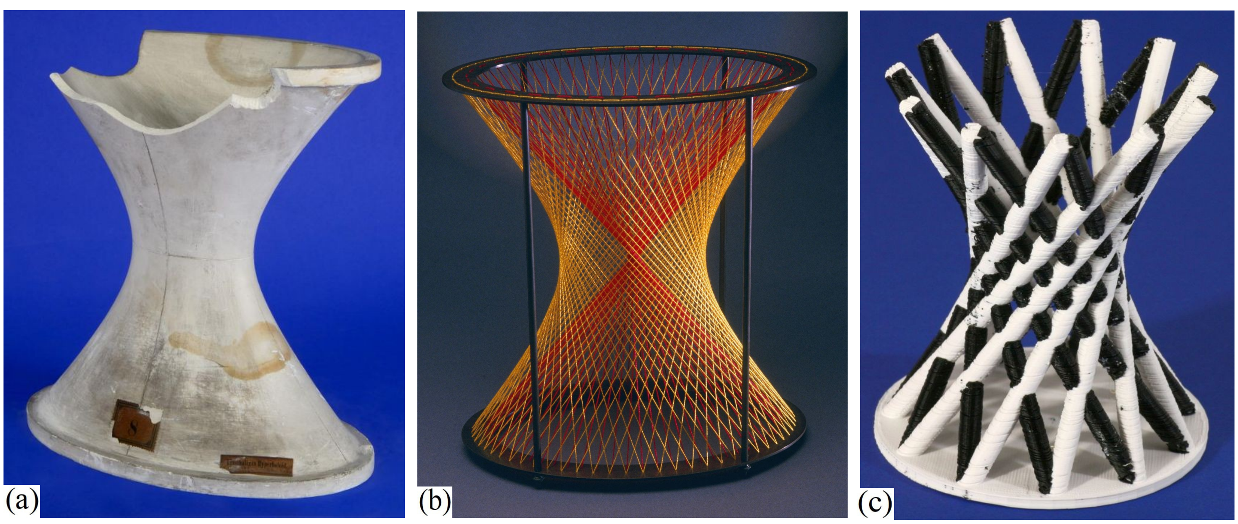


Figure 2: *Models: (a) Plaster surface. (b) Strings in metal frame. (c) 3D-printed plastic.*

The ingenious Russian engineer Vladimir Shukhov (1853-1939) was the first to apply stick hyperboloids to architectural-scale constructions, patenting the idea in 1895. [5, 6] Like Wren, he was inspired by a woven basket. Starting in the 1880’s, he made a career of designing and building roughly 200 large water towers, radio towers, light houses, etc., that made very efficient use of materials with stick hyperboloid structures. Figure 3(a) shows Shukhov’s first example, a 37-meter tall water tower at the All-Russia Industrial and Art Exhibition of 1896. Then in the first half of the twentieth century, stick hyperboloid towers of Shukhov's design were used for ship masts, first on Russian military ships, and then soon on US battleships until World War II. See Figure 3(b). [2] Further inspired by Shukhov's works, contemporary architects have gone on to build impressive modern examples such as the 108 meter tall Kobe Port Tower in Kobe Japan, Figure 3(c). Many examples of buildings and towers based on stick hyperboloids are collected on a Wikipedia page. [17] As far as I can determine, almost all architectural-scale examples have circular horizontal cross sections, but the Canton tower, a 600 meter tall observation tower in Guangzhou, China, has elliptical cross sections. The rotation axis is typically oriented vertically, but the lovely Corporation Street Bridge (Figure 3(d)) is a horizontal hyperboloid structure connecting buildings across a street in Manchester, England.

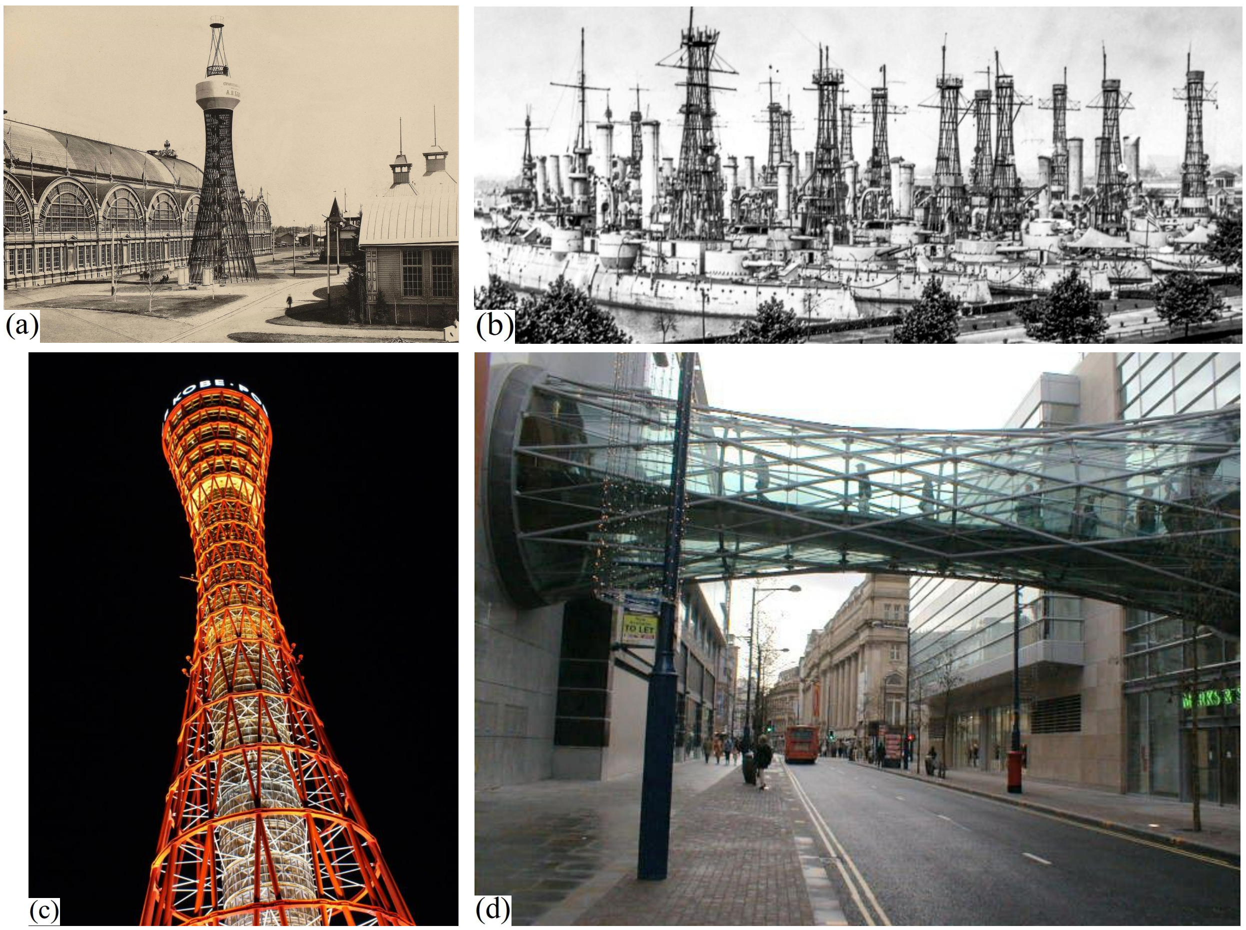


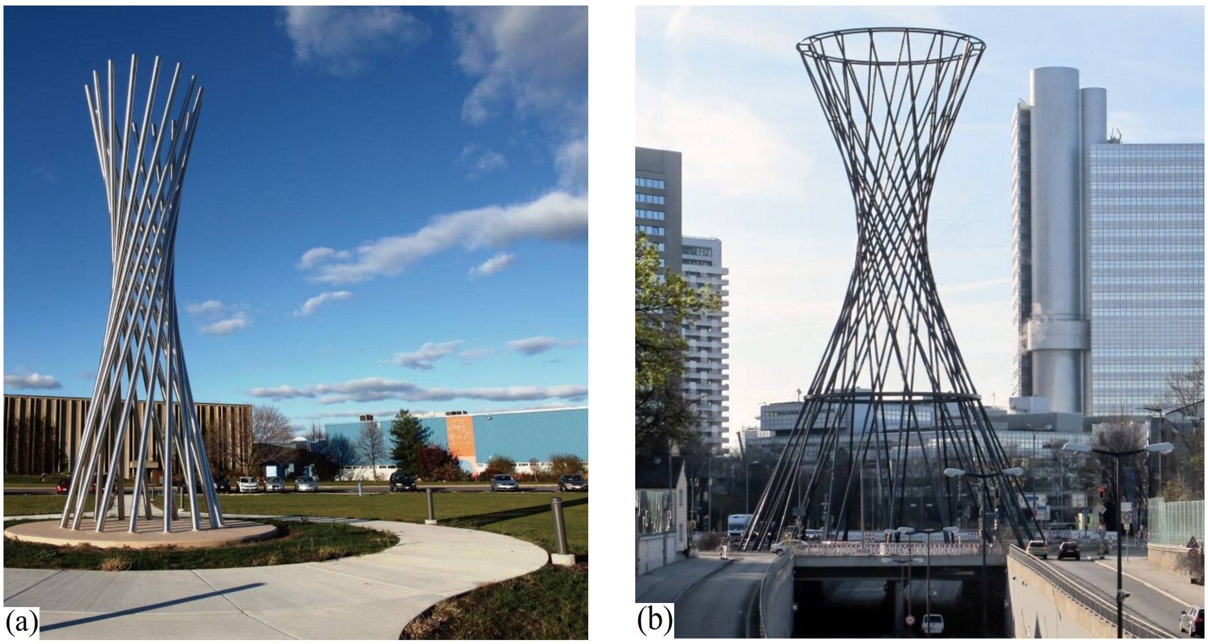
Figure 3: A*rchitectural examples of stick hyperboloids: (a) 1896 tower by Shukhov; (b) battleship masts; (c) Kobe Tower; (d) Corporation Street Bridge.*

On an indoor scale, as a common example of minimalist geometric design, one often sees stick hyperboloids worked into the structure of furniture, such as the bases of wicker tables and chairs. A search of designer housewares websites reveals a variety of waste-paper baskets, fruit bowls, hampers, and furniture items that are essentially stick hyperboloids. The baskets that Wren and Shukhov mention as inspirations must have been similar.

Some mathematically aware artists have appropriated the stick hyperboloid form for aesthetic ends. A large, purely sculptural example is the 12-meter tall *Tracticious* designed by Robert

Wilson for Fermilab, in Batavia, Illinois. (Figure 4(a)) [7] It was created in 1988 from 6.5-inch diameter stainless steel cryostatic tubes that were left-over from constructing the Tevatron's superconducting magnets. The sculpture displays only the right-handed family of rulings, with the rods anchored in the cement base to position them so they do not contact each other.

The largest stick hyperboloid I know that was constructed purely as art is the 52-meter tall *Mae West* sculpture in Munich, Germany, completed in 2011. (Figure 4(b)) [3] The material is carbon-fiber reinforced plastic. Its large size (a tram line runs through it), cost, and form were politically controversial, but clearly the artist, Rita McBride, made dramatic use of new technology that can manufacture very long continuous lengths of the strong lightweight tubing.



(a) (b)

Figure 4: *Sculptural examples of stick hyperboloids: (a) Tracticious; (b) Mae West.*

Inspired by the elegance, history, and applications of hyperboloids, I have long thought about hands-on construction workshops to make more people familiar with them and with their mathematical characteristics. The first program I ran of this sort was at a Saturday morning math enrichment program at Brookhaven National Laboratories in 2010, where students each made the skewer and rubber-band model shown in Figure 1(b). [8] A few years later, Elisabeth Heathfield and I expanded this skewer (or chopsticks) workshop to culminate in a larger-scale version made from 4-foot wooden dowels. We wrote it up in a detailed three-part lesson plan that is part of our Making Math Visible project. [12] After leading the activity at many sites, it is satisfying to see it becoming a popular math construction activity, with many variations posted online by others.

Elisabeth and I have also made garden trellises from five-foot long bamboo stakes held together with cable ties. (Figure 5(a)) The largest stick hyperboloid I have made is a ten-foot arbor-way, designed as a means to bring mathematics into a teaching garden. I led a group of students and faculty in its construction in the summer of 2014, while I was artist in residence at the University of British Columbia in Vancouver. Shown in Figure 5(b), it consists of forty 12-foot long bamboo poles (twenty in each direction) held together with heavy rubber bands. It has an elliptical cross section and is oriented horizontally to walk through. A construction video details the entire process. [9]

(a)  (b) 

Figure 5: *(a) Five-foot tall garden trellis. (b) Eight-foot garden arborway at UBC.*

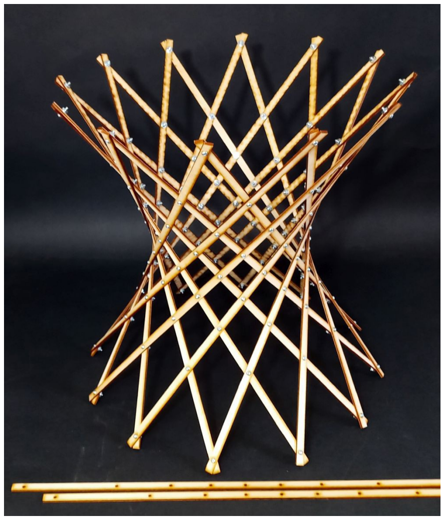
On a similar scale (but non-rigid), in 2011 while designing the Museum of Mathematics in New York City, I created a dynamic hyperboloid exhibit large enough for visitors to sit inside of and spin around. The upper circle is fixed to the ceiling, but the lower circle rotates, twisting elastics from a cylinder through a continuum of hyperboloids to a double cone and back. I first built a four-meter tall string-and-plywood prototype in the unfinished museum space as a maquette of the concept before having professional fabricators build the visitor-ready version. [10]

A final observation about these examples is that one can flatten a skewer-and-rubber-band hyperboloid like a pancake or squeeze it into a narrow tube. Students love this because it can be flattened on a table and suddenly released, to spring up into the air. The video [9] illustrates the transformation. This only applies to Figures 1(b) and 5(a), as all the other structures above are triangulated or rigidified in some way to eliminate this flexibility.

Three Questions about Stick Hyperboloids

There is an extensive literature going back centuries concerning hyperboloids, looking at them from mathematical, architectural, and educational viewpoints (among others). I have searched, yet I have not seen any discussion of three basic questions that naturally come up when thinking about the geometry of stick hyperboloids. To appreciate the questions, first consider the model shown in Figure 6. It is a stick hyperboloid made from 24 identical two-foot-long laser-cut strips of 3mm thick plywood (12 in each direction) with holes to receive small nuts and bolts. Two unused individual pieces are also shown, emphasizing that the holes can be made before assembly and they are not evenly spaced. Note that the spare pieces are completely flat, just as they come out of the laser cutter, but in the hyperboloid structure each piece has a significant torsional twist along its length. With all this background, now consider these questions:

1) What is the spacing pattern of the crossing points? (Or, how should one plan the stick length and hole positions for a given hyperboloid as target shape?)
2) How does each strut twist along its length? (Or at what angles should the holes be drilled if the rod is a cylindrical dowel?)
3) What happens if the structure is flattened to a pancake or squeezed into a cigar shape? Is there slippage at the connection points in the physical model or is a mathematically exact motion possible as an ideal linkage?



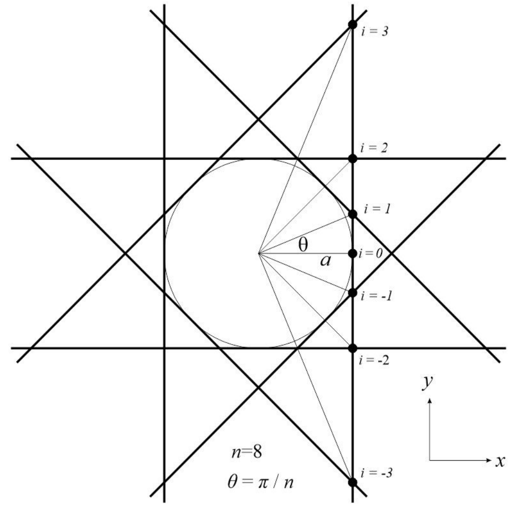


Figure 6: *Stick hyperboloid made from twisted slats.* **Figure 7:** *Sticks projected.*

Engineers making large-scale stick hyperboloids, such as Vladimir Shukhov, must certainly have thought about these questions and worked out the answers for themselves, but I don't see that anyone has recorded the solutions in any accessible reference. This section addresses these three questions, so is a bit more technical with some trigonometry and analytic geometry. I include dimensions in order to design to a specific size and verify all equations are dimensionally consistent. Begin with the general form $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$, where the variables and parameters have units of length. Using m for meter, the major and minor radius parameters a and b might be 1m and 0.5m for the elliptical garden hyperboloid of Figure 5(b). They specify the cross-section at z = 0m. But for simplicity, I consider only the common circular case, i.e., $b = a$. So drop b to get the two-parameter equation: $(x/a)^2 + (y/a)^2 - (z/c)^2 = 1$. This hyperboloid is asymptotic to the circular cone (visible in Figure 2(b)): $(x/a)^2 + (y/a)^2 - (z/c)^2 = 0$, which has slope $\pm c/a$, (thinking of z as vertical). The ruling lines also have this same slope, $\pm c/a$, as can be seen in Figure 2(b) or worked out by setting $x = a$ in the hyperboloid equation to take a cross sectional slice, and then solving for z/y.

Let n be the desired number of struts in each direction. If one wishes to calculate the sticks' crossing locations, the first step is to choose parameters a and c. Set a as the radius at the waist where z = 0m. One can then play with c in graphing software to find an attractive shape or solve for c by specifying the desired radius r at any height h, so $c = h/\sqrt{(r/a)^2 - 1}$.

Hole Spacing: First project the sticks to the xy plane, as in Figure 7, which shows the case of n=8. Each heavy line represents two coplanar sticks (one right-handed, one left-handed). The circle of radius a is the waist of the hyperboloid, at z=0m, to which all the projected sticks are tangent. Seven crossing points in the x=a plane are shown as heavy dots; one indexed with i=0 is at the stick midpoint. The radial lines are spaced with an angular separation of $\theta = \pi/n$. The distance in the xy plane from the midpoint to crossing i is $a\tan(i\theta)$. That distance is scaled by the slope c/a to get the vertical distance to the point. Then the Pythagorean theorem gives the distance along the stick. Working this out, the position to drill hole i is $\sqrt{a^2+c^2}\,\tan(\pi i/n)$. I used this formula to determine where to laser-cut the bolt-holes for the struts in Figure 6. For odd n, each stick crosses the n sticks of the other handedness. For even n, each crosses only n-1 others, as one is parallel. Either way, the furthest values of i are $\pm$floor((n-1)/2).

Strut Torsion: When making stick hyperboloids from cylindrical rods and rubber bands as in Figure 1(b) or 5(a), one may be oblivious to the twisting relationship between the surface and each stick. Each crossing's contact point is at a slightly different angle (relative to any ruling line of the rod). If one used rigid 2×4s and drilled parallel bolt holes, it would be obvious that the sticks could not be assembled into a curved surface. The sticks or the holes must twist along their length, as is visible in Figure 6. Looking at a side view of a ruling line relative to the asymptotic cone, as in Figure 2(b), it should be clear that this torsion changes by a full 180 degrees along the infinite length of the line. To quantify it exactly, one can look at how the surface normal varies as a function of distance along a typical ruling line, e.g., the line through the midpoint (a, 0, 0) that goes in the (0, 1, c/a) direction. The normal at any point of the surface is in the gradient direction: (c^2x, c^2y, $-a^2z$). Knowing that the dot product of two unit-length vectors is the cosine of the angle between them, one can solve for that angle. After a few messy steps (because distance and normalization introduce a square root) and a small trig manipulation (replacing the arccosine with an arctangent) everything simplifies beautifully. Details are in the online supplement to this paper. The result is that if d is the distance along the stick measured from the midpoint, the torsion angle is simply $\psi = \arctan(d/c)$. For small c (giving a pancake shape), the flip happens quickly and the torsion quickly saturates at ± 90 degrees. For large c (a cigar shape), ψ undergoes small linear steps from crossing to crossing, as with a helix.

Flexing: Stick hyperboloids with suitable joints at the crossings do flex. Courant and Robbins briefly mention this [4, p. 214] and Hilbert and Cohn-Vassen give a detailed analysis [14, p. 29] but neither discuss the change in torsion. It is easy to understand the flexing by uniformly scaling Figure 7: as distances in the xy plane change, the stick slope must change accordingly for the projection to work, but the crossing points along the stick can remain unchanged (up to the limiting point where the sticks have zero slope). Hilbert and Cohn-Vassen approach it differently and give a lengthy proof that flexing preserves distances, but they also go further and show that all the stages in the flex are "confocal" [18], meaning if you slice the hyperboloid on a plane through the z axis to reveal a hyperbola cross section, its two foci are fixed throughout the flexing. Formally, to preserve the foci, a^2 and c^2 change to $a^2-\lambda$ and $c^2+\lambda$, respectively, which preserves the formula above for the hole positions, but changes the value for the torsion.

The results above show that flexing requires the torsion at each point to change (because c is changing and ψ is a function of c). So flexing preserves the hole spacing but not the stick torsion. This assumes an unusual type of connecting joint: a kind of pass-through ball and socket joint. With ordinary pin joints, like the bolts in Figure 6, which allow for rotation but not torsional change, a stick hyperboloid can not truly flex. Conveniently, the rubber band joints in a skewer hyperboloid like Figure 1(b) permit slippage, so they do flex in practice.

Hyperboloid Woven from Paper Strips

As an offshoot to this analysis I designed a new form of paper model. Figure 8(a) shows a triangulated approximation to a hyperboloid surface made by a simple weaving of two families of paper strips. The edges of each strip lie on two adjacent ruling lines. In this example there are twelve yellow right-handed strips (Figure 8(b)) and twelve red left-handed ones. The template for them is shown in Figure 8(c); the dotted lines are fold lines where a slight zig-zag crease should be made before weaving everything together. A pair of paper triangles connects via a crease to cover each (non-planar) kite-shaped opening of the implicit stick hyperboloid. Clear tape at the top and bottom rims connect the inner and outer layers. The edge lengths for the triangulated template can be calculated using the hole spacing formula above.

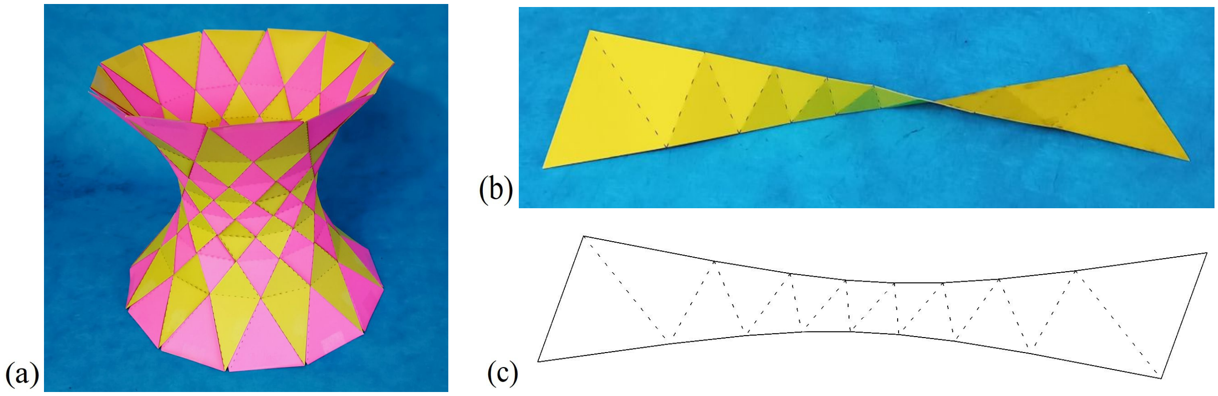


Figure 8: *(a) Woven paper hyperboloid ; (b) one pre-creased laser-cut strip; (c) template.*

Conclusion

Inspired by the storied history of hyperboloid surfaces in the mathematical, educational, architectural, and artistic literature, I have designed and built a variety of original hyperboloid forms with sticks and with paper. Two original results presented here are useful when pre-drilling sticks: (a) for an n-stick hyperboloid model with parameters a and c, the position to drill hole i is $\sqrt{a^2+c^2}\tan(\pi i/n)$ and (b) the torsional angle to drill a hole at position d (relative to the hole at the midpoint) is $\psi=\arctan(d/c)$. A simple method of understanding what happens when a stick hyperboloid is flexed clarifies that the connection spacing need not change but there is torsional rotation along each stick changing the directions of the connections as the structure is flexed. There is undoubtedly much more to explore about stick hyperboloids as they are endlessly fascinating with their beautiful blend of curved and straight.

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