The Ticking Clock: A Geometric Interpretation of Modular Multiplicative Inverses

Stephen E. Erfle

International Business and Management, Dickinson College, Carlisle PA; erfle@dickinson.edu

Abstract

This paper introduces users to a class of images created using the *Electronic String Art* model which was developed to encourage exploration of the relation between patterns: geometric, numeric, and artistic. These *60-second images* are created using 60 identical subimages in which each subimage ends one vertex beyond the previous vertex, just like the second hand on a clock. This one vertex by one vertex clockwise movement is the geometric consequence of having two of the parameters being modular multiplicative inverses.

Introduction and Motivation

At one time or another, all of us have found ourselves looking at a clock, entranced by the simple second by second movement of the second hand as it clicks around the clockface. This idle introspection led me to analyze a class of images I call *60-second images*. One can see a 60-second image emerge as a result of 60 identical subimages, each of which ends one vertex farther along the polygon than where it started, just like with the second hand of a clock. Figure 1 shows three 180-line 60-second images with the first 1/60th shown in red. A blue letter beneath an image is a link to a version of [4] set to that image. Alternatively, you can open [4] and adjust the parameters to those given below the image and use *Fixed Count Line Drawing (FCLD)* mode to generate the image line by line.

One can enjoy exploring 60-second images (and other images) without being concerned with how the images are created. Put another way, this paper will be of interest to those who just want to play, but the goal here is to explain why the images are created in this way. To accomplish this, we first examine the parameters underlying the *Electronic String Art* model [2], *ESA*. Next, we examine the attributes that produce 60-second images and explore some of the images possible in this situation. Finally, we present variations on this theme.

| (a) $P = 97, J = 13$ | (b) $P = 29, J = 29$ | (c) $P = 89, J = 29$ |
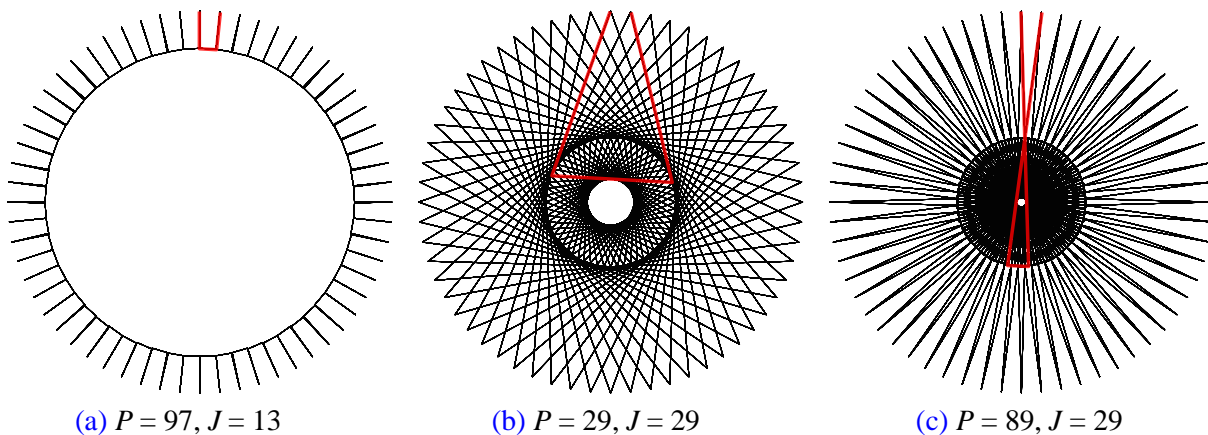
Figure 1: *Examples of 60-second images with n = 60 and S = 3 (with first three lines shown in red).*

Exploring the Parameters Controlling the *Electronic String Art* Model

The *ESA* model uses the ideas set forth in Erfle and Erfle [1] with one important difference: rather than allowing users free reign in setting vertices in the (x, y)-plane, vertices are restricted to those of a regular

polygon. This simplifies the model significantly because four parameters control the image and each parameter can be controlled via up and down scroll arrows, ⬍. This makes the model accessible to younger (or less mathematically sophisticated) users because all they need do is click and watch what happens. Scroll arrows allow users to rapidly adjust values and quickly test hypotheses even if those hypotheses are incompletely understood. I argue that such purposeful play can lead to learning, even if that learning is informal in nature [2], and is consistent with Stein's *Triex: Explore, Extract, Explain* [7] view of how mathematics should be taught.

I provide a short overview of the landscape detailed in [2]. The *ESA* model is controlled by four parameters, n, S, P, and J which can be thought of in two groups: n and J, and, S and P. The first two create the *vertex frame*, VF, shown in blue in Figures 2(a) and 2(c). The second two set the number of subdivisions and create the counting rule for drawing the image. Note that all equations discussed in this paper are in Column M of the *String Art MMI Excel* file which is provided as a supplement to this paper.

Definition. The *vertex frame*, VF, is the polygon or star on which all subdivision endpoints are located.

The VF is a *continuously-drawn* star using the n vertices of a regular n-gon where there are J vertex jumps between successive lines of the star, denotes as an n,J-star. For example, there are two distinct stars for $n = 7$: 0–2–4–6–1–3–5–0 if $J = 2$ (the – means connect vertices in the frame), and 0–3–6–2–5–1–4–0 if $J = 3$. When $J = 1$ or $J = n$-1, the VF is a polygon. Vertices are enumerated *clockwise*, CW, starting at the top (0) so that the only difference between these polygons is the order in which sides of the VF are drawn (CW for $J = 1$, *counterclockwise*, CCW, for $J = n$-1). The same is true for $J = J_0$ and $J = n$-J_0 so that we can restrict our analysis to $J < n/2$ if we only care about distinct images instead of how they are drawn.

If $1 < J < n/2$, the VF will be an n-point star if n and J are coprime. The *Vertex Common Factor, VCF,* is VCF = GCD(n, J) where GCD is the *Greatest Common Divisor*. If n and J are not coprime, the VF uses only n/VCF vertices, so that there is only one <u>distinct</u> 12-point star ($J = 5$) but there are four 11-point stars ($J = 2, 3, 4, 5$). Three points are worth noting: First, only *continuously-drawn* images are considered here (when $n = 6$ and $J = 2$, the VF is a triangle); Second, the only difference in VF between J and n-J is the direction in which the VF is drawn (compare Figures 3(a) and 3(b)); Third, all endpoints of lines in the image are on the VF, shown as the blue square in Figure 2(a) and blue 5,2-star in 2(c). The vertices of the $n = 4$ square and $n = 5$ pentagon are the red numbers 0 to 3 in 2(a) and 0 to 4 in 2(c).


(a) $n = 4$, $J = 1$ (b) *8-point star in square*   (c) $n = 5$, $J = 2$ (d) *5-point star in star*
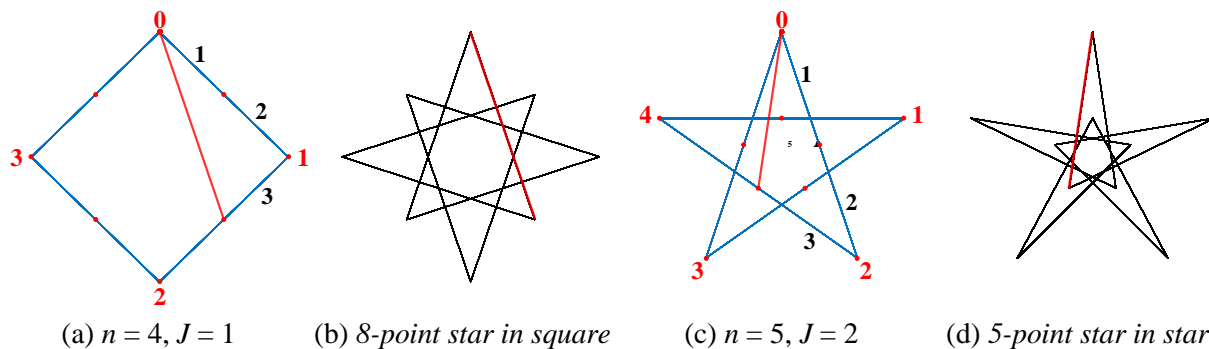
Figure 2: *Building two ESA images with S = 2 and P = 3. Vertex frame in blue and first line in red.*

Next, create S equal subdivisions on each VF line ($S = 2$ in Figure 2). This produces $S \cdot n$/VCF subdivision endpoints (8 red dots in 2(a), 10 in 2(b)). The image is created from these endpoints using rules (a) – (e).

    (a) The top of the image is the start of the first line in the image. As noted above, this is vertex 0.
    (b) Count P subdivisions on the VF (black numbers 1, 2, 3 in Figures 2(a) and 2(c) given $P = 3$).
    (c) Draw a line from the top to Pth subdivision endpoint (the red line in Figures 2 and 3).
    (d) Count out P more subdivisions and draw a line from the end of the previous endpoint to the new endpoint.
    (e) Repeat (d) until the endpoint completes the circuit and is once again at vertex 0 in the figure.

The final image will use all subdivision endpoints if P and $S{\cdot}n$/VCF are coprime. Total number of lines is $Lines = (S{\cdot}n/\text{VCF})/SCF$ where $SCF = \text{GCD}(S{\cdot}n/\text{VCF}, P)$ is the *Subdivision Common Factor* (in Figure 2(b), Lines = 8 and in 2(d), Lines = 10 since SCF = 1 for both). Based on Figure 2, what would you change to get a *10-point star in pentagon, 14-point star in heptagon,* or a *7-point star in star*? Check your answers with the *String Art MMI Excel* file or [4].

Figure 3 examines the consequence of changing three of the four parameters, one at a time. Because $n = 3$, the VF is a triangle regardless of J but changing J from 1 to 2 in moving from 3(a) to 3(b) reverses the order that the VF is drawn (CW for panels 3(a), 3(c), and 3(d); CCW for panel 3(b)) and therefore changes the order in which the lines are drawn in 3(a) and 3(b). Comparing 3(a) and 3(c) shows what happens when S changes from 3 to 5 (when $S = 4$ the image is the triangular VF). When $S > P$ at least part of the VF is included in the image as we see in 3(c), 3(d) and 4(a), but not if $P > S$ (like in 3(a) and 3(b), as well as 1, 2(b), 2(d), 4(c) and 4(d)). Both 3(a) and 3(d) require 9 connected line segments to create the image, but those segments are more apparent in 3(a) than 3(d). After all, it looks like you could draw 3(d) with 6 segments, but this is only true if you lift your pencil (meaning draw disconnected segments). *ESA* images are created by following a counting rule using connected (*continuously-drawn*) segments.


| (a) Base Image | (b) Change J to $J = 2$ | (c) Change S to $S = 5$ | (d) Change P to $P = 2$ |
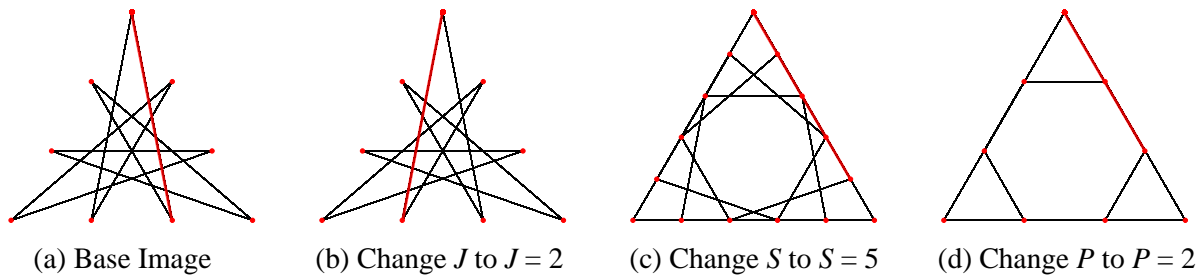
Figure 3: *Varying one parameter at a time given Base Image values of $n = 3$, $J = 1$, $S = 3$, and $P = 4$.*

Vertex-to-Vertex Cycle Generators

The final image is created as a closed series of connected line segments. At least one of the endpoints must be the vertex of the parent polygon (because 0 is always an endpoint), but other polygonal vertices may be excluded from the final image as Figures 4(a) and 4(c) show. If more than one vertex is used in the final image, then that image will have rotational symmetry equal to the number of vertices used. This means that to understand the image, one need only focus on the part of the image that is created from one vertex to the next time a vertex is used. We call this part of the image a *vertex-to-vertex cycle generator.*

In this context, a vertex-to-vertex cycle generator, *CG*, is a subimage. The image is composed of a number of CGs rotationally concatenated to one another, the last of which completes the cycle by ending at the starting point (vertex 0 in *ESA* models).
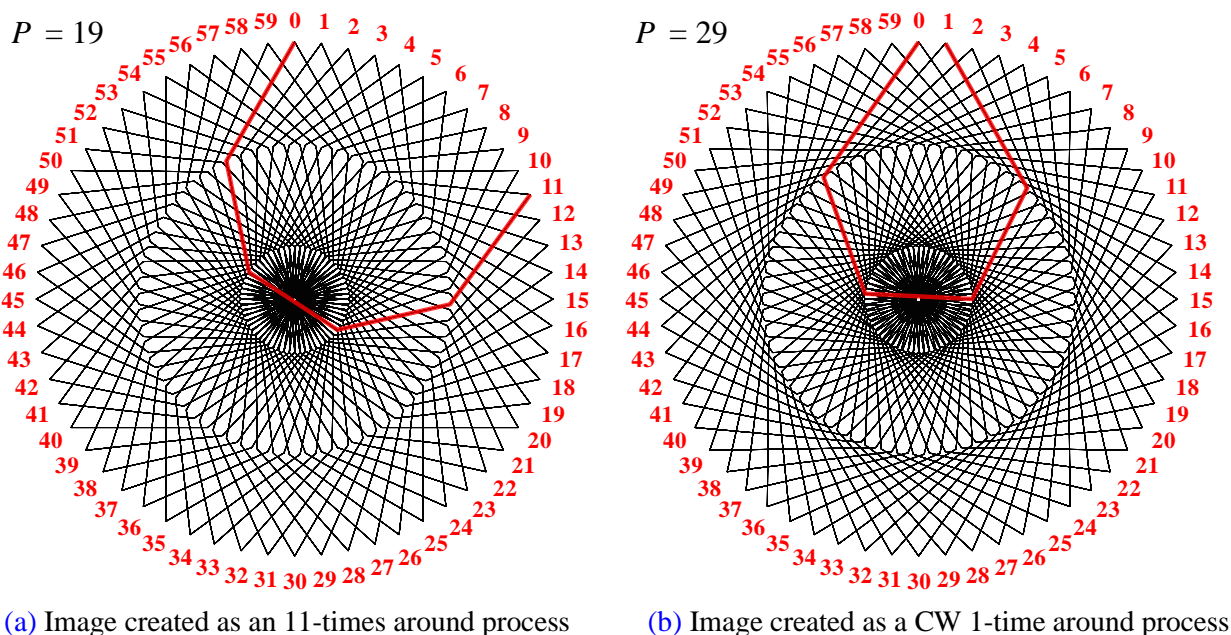

| (a) $P = 2$, SCF = 2 | (b) $P = 3$, SCF = 3 | (c) $P = 4$, SCF = 4 | (d) $P = 5$, SCF = 1 |
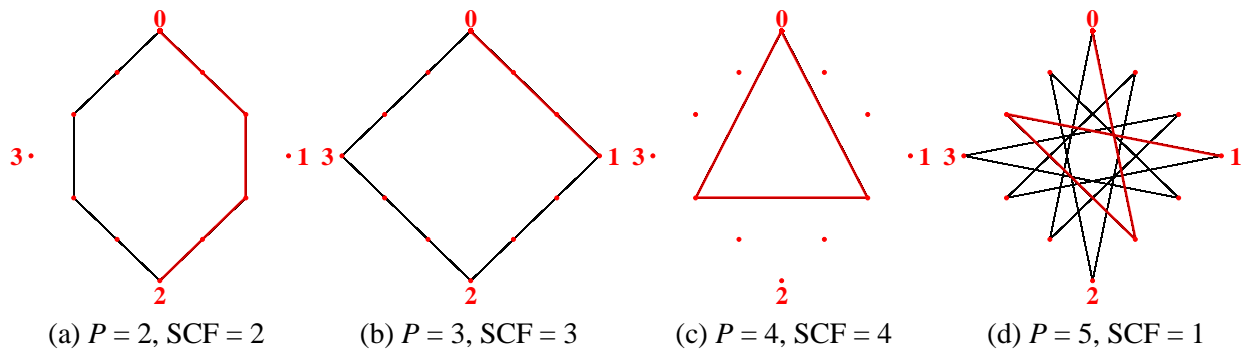
Figure 4: *Images with $n = 4$, $S = 3$ and $J = 1$ showing first cycle generator, CG, in red.*

The first CG ends at vertex 2 in Figure 4(a), 1 in 4(b), 0 in 4(c), and 1 in 4(d). Note that the fraction of subdivision endpoints used in each image is 1/SCF. The number of lines per CG, C, is $C = S/GCD(S, P)$ and the number of CGs in the image, M, is $M = $ Lines/C. Figure 4(a) has two 3-line CGs, 4(b) has four 1-line CGs, 4(c) has one 3-line CG, and 4(d) has four 3-line CGs.

60-Second Images and Modular Multiplicative Inverses

We are now able to analyze 60-second images. We require 60 vertices to be used so we restrict our analysis to $n = 60$ and J coprime to 60, VCF = GCD(60, J) = 1. Additionally, we want S to be coprime with P so that CGs have S lines per CG and a total of $60S$ lines in the image. Figure 5 provides two candidate images with $S = 5$ and $J = 29$ with links to the web version [4] at labels (a) and (b). If the first CGs were not shown in red, you would be hard-pressed to see a difference between the two images.

(a) Image created as an 11-times around process   (b) Image created as a CW 1-time around process

Figure 5: *Comparing two n = 60, S = 5, J = 29, 300-line images with first CG in red.*

Consider how Figure 5(a) is drawn. The first CG ends at 11 so subsequent CGs end at 22, 33, 44, 55, 6, 17, … (6 = 66 MOD 60 and 17 = 77 MOD 60). Note that each time "around" uses one more of the vertices from 1 to 10 that were skipped because the end of the first CG was 11. The image appears to be a rotating 5-petal flower (thanks to Liam Myles [4], you can see this image emerge by clicking the link at (a) and then click *Drawing Mode, FCLD* with *Drawn Lines, DL = S* = 5) that is completed after 11-times around once all vertices from 1 to 10 are filled in with the last line connecting to the first at vertex 0 at the end of sixty 11-vertex-jump CGs. By contrast, 5(b) is completed as a CW 1-time around process because each "pentagonal" CG ends one vertex after the prior CG until the image is complete after sixty 1-vertex-jump CGs when viewed dynamically via the link at (b).

You may have noticed that 5(b) is the second 60-second image shown with $J = 29$ and $P = 29$ (the first being 1(b) where $S = 3$). Additional versions occur as long as GCD(S, P) = 1 because one CG is S lines or $P \cdot S$ subdivisions, but every S subdivisions is a vertex that is J away from the prior vertex so that P jumps of J is the ending vertex relative to the starting vertex. The ending vertex is 1 larger than the starting vertex if $P \cdot J$ is 1 larger than a multiple of 60. Put another way, *P and J are multiplicative inverses modulo 60*.

Definition. When numbers a, b, and m have the property that $a \cdot b = k \cdot m + 1$ (for some k) we say that a and b are modular multiplicative inverses modulo m (a and b are MMI MOD m iff 1= $a \cdot b$ MOD m).

Modular multiplicative inverses have been used to produce music [5], create art [6], as well as in cryptography [3]. Table 1 shows the 16 values of *J* that are coprime with 60 and the smallest paired values of *P* producing 60-second images. These pairs are not as hard to find as you might imagine because $P \cdot J$ must have a last digit of 1. This requires that the last digits of *J* and *P* are both 1, or both 9, or that one has last digit 3 and the other 7. If two numbers *b* and *m* are coprime, then one can backtrack Euclid's Algorithm for finding GCD to determine *a*, the MMI of *b* MOD *m*. The *String Art MMI Excel* file contains an additional worksheet for finding MMI called *Backtracking Euclid* that is based in Section 24.3 of [2].

Table 1: *Jump, J, and smallest subdivisions between points, P, pairings producing 60-second images.*

J	1	7	11	13	17	19	23	29	31	37	41	43	47	49	53	59
smallest *P*	1	43	11	37	53	19	47	29	31	13	41	7	23	49	17	59

Additional *P* possibilities exist for a given *J*. Distinct 60-second images occur for $P+60 \cdot k$ for $0 \leq k < S$. Figures 1(b) and 1(c) are the $k = 0$ and $k = 1$ versions of $J = 29$ and $P = 29$. You should verify that $k = 2$ (i.e., $P = 149$) produces a third distinct image, but $P = 209$ (i.e., $k = 3$) is an identical image to $P = 29$. The reason for this identity is straightforward: there are $S \cdot n = 180$ possible subdivisions so counting 209 is the same as counting all of them once then the first 29 a second time.

Spinning Stars, Pulsing Polygons, Donuts, and Stacked Circles

As you play with the model, you will find a wide array of 60-second images. Some of the images are similar to those found in [2] although those images were typically based on smaller *n* than 60. Figure 6 shows some of the possibilities with links to the image on the companion website (use *FCLD* mode in [4] with $DL = S$ to see each image emerge in 60 1-vertex-forward CGs). This list is by no means exhaustive, and other examples are readily available (for example change $S = 24$ which produced 6 pulsing rectangles in 6(b) to $S = 27$ and see 9 pulsing triangles or $S = 30$ to see 6 pulsing pentagrams). By changing *J* (and hence *P*), additional images emerge like this (60, 14, 29, 29) chrysanthemum. (The *String Art MMI Excel* file and the companion website [4] will show greater detail than possible in Figure 6 due to the size constraints here.)
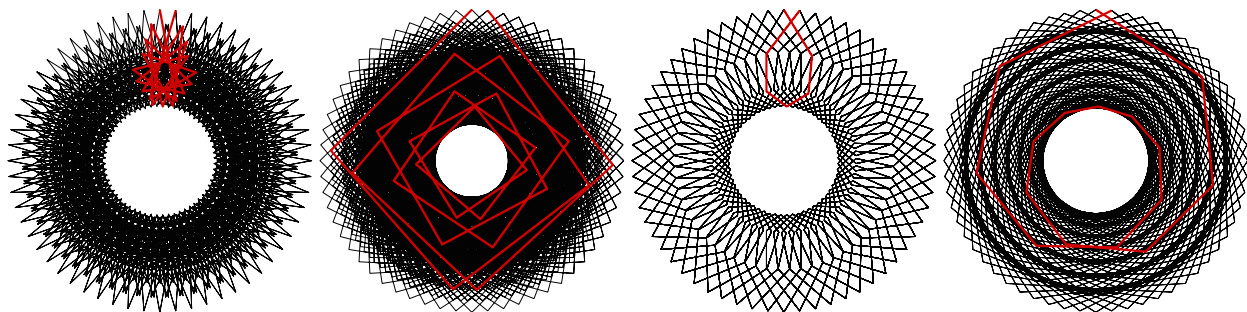
(a) *Spinning Stars* ($S = 18$)   (b) *Pulsing Polygons* ($S = 24$)   (c) *Donuts* ($S = 6$)   (d) *Stacked Circles* ($S = 16$)

Figure 6: *Examples 60-second image types using n = 60, J = 23, and P = 47 with first CG in red.*

One final point to make applies to all of the images discussed above. As noted in the discussion of Figures 3(a) and 3(b), the *ESA* model produces images with vertical symmetry. Suppose you find a CW 60-second image you like based on $(60, S_0, P_0, J_0)$. The image $(60, S_0, P_0, 60\text{-}J_0)$ will look the same as the initial image. But if you click *Drawing Mode* then choose *FCLD* with $DL = S_0$, you will find that the image is now a CCW one-time around image. Put another way, if you start with a CW 60-second image and change *J* to 60-*J*, you can turn back time!

Variations on a Theme

The 60-second images discussed above require an $n = 60$ image that is created one CG at a time in a CW 1-time around fashion because the first CG ends at vertex 1. Each CG is identical to the prior CG. One can obtain 60 recognizable subimages using alternative methods that are also based on MMI.

60 Polygons and Stars

In Section 10.5 of [2], I offer rules for creating polygons and stars in a CG for general n. Start with an image with recognizable subimages such as Figure 7(a) with 2 hanging quadrangles per CG. This image requires two applications of MMI, one to assure that it is 1-time around, the other to assure that the subimages are formed as a single-step process. Both processes require that we consider the product of numbers that are 1 less than a multiple of another as well as MMI.

Definition. When numbers a, b, and m have the property that $a \cdot b = k \cdot m$-1 (for some k) we say that a and b are negative modular multiplicative inverses modulo m (a and b are nMMI MOD m iff m-1 = $a \cdot b$ MOD m).

Definition. An image is a CW 1-time around image with n CGs if P and J are MMI MOD n. An image is a CCW 1-time around image with n CGs if P and J are nMMI MOD n.

Definition. An image is *single-step of length L* if $L \cdot P = k \cdot n \cdot S \pm 1$. Put another way, an image is single-step of length L if $L \cdot P$ is MMI or nMMI MOD $n \cdot S$.

To understand these definitions, consider a simple comparison: (4, 10, 13, 1) and (4, 10, 27, 1). If the first is drawn using *FCLD* with $DL = 3$, it will appear to be a CCW rotating triangle since $3 \cdot 13 = 4 \cdot 10$-1 or 1 less than a multiple of $n \cdot S$. The second will appear drawn as a CW rotating triangle since $3 \cdot 27 = 2 \cdot 4 \cdot 10$+1 or 1 more than a multiple of $n \cdot S$. In both instances, the image will appear drawn as a series of single-step triangles. Note that these two examples produce the same static image. The only difference is the way the image is drawn (since $27 = P_2 = n \cdot S$-P_1, just like Figures 3(a) and 3(b) which compared J and n-J).

Given these definitions, 7(a) is single-step of length 4 since $L \cdot P = 4 \cdot 34 = 136 = 15 \cdot 9$+1 = $n \cdot S$+1, and it is also CW 1-time-around since P and J are MMI MOD n. To create additional repetitions of the subimage and turn 7(a) into 7(b) and 7(c) consider the following rule: For fixed $n = 15$ and $J = 4$, add 4 to S and 15 to P. More generally, to add one more MINIMUM(J, n-J)-gon or MINIMUM(J, n-J)-gram per CG simply add MINIMUM(J, n-J) to S and n to P. This works as long as the new image has SCF = 1.
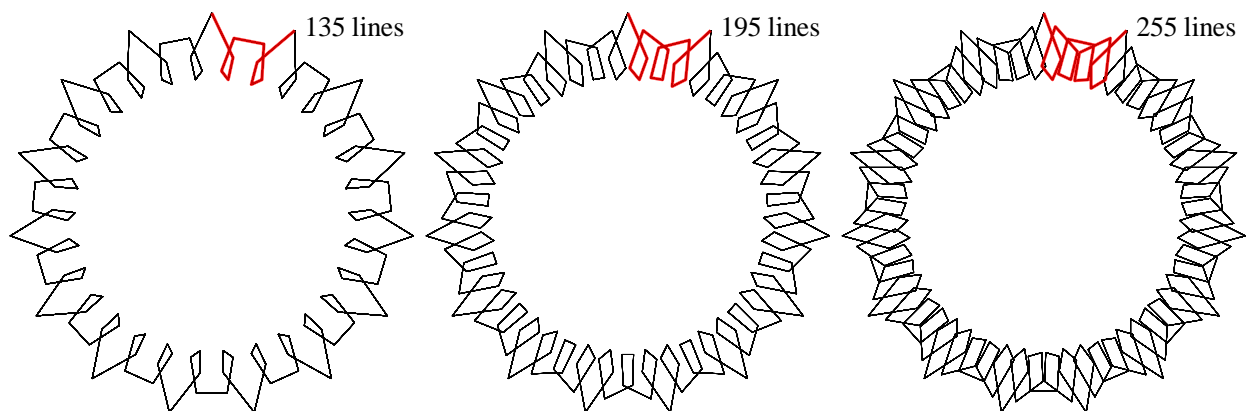
(a) (15, 9, 34, 4) 2 4-gons per CG   (b) (15, 13, 49, 4) 3 4-gons per CG   (c) (15, 17, 64, 4) 4 4-gons per CG

Figure 7: *A progression of linked images based on Polygons and Stars in a CG (first CG in red).*

The progression in Figure 7 was chosen because 7(c) has 60 quadrangles. The four quadrangles in the first CG are not identical to one another but are recognizable. If n is a factor of 60 and the number of rotating polygons or polygrams in the first CG is 60/n, each image will have a total of 60-gons or 60-grams. Figure 8(a)-(c) provide three such examples and more are available at Section 13.2 of [2].

324 lines   288 lines   190 lines
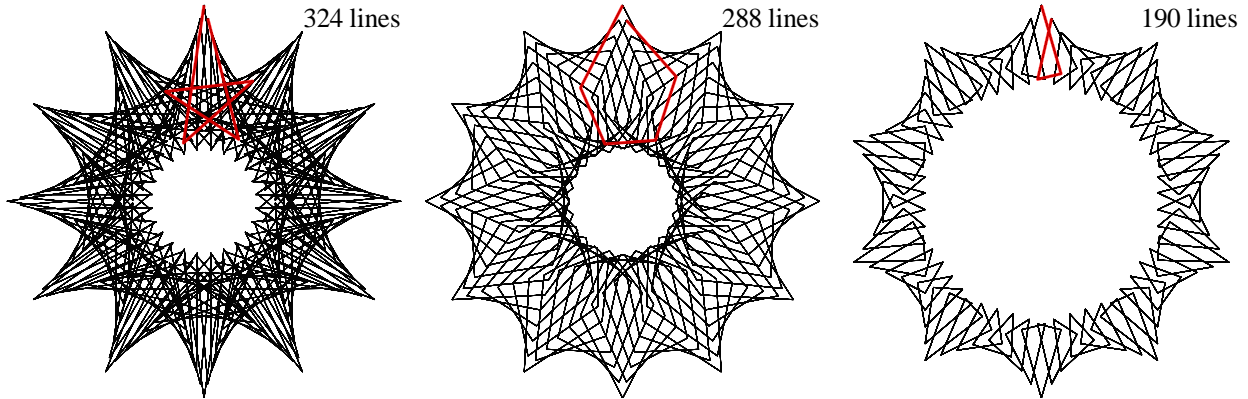
(a) (12, 27, 65, 5) 5 pentagrams/CG (b) (12, 24, 115, 7) 5 pentagons/CG (c) (10, 19, 127, 3) 6 triangles/CG

Figure 8: *Examples of 60 polygon or polygram images with first polygon or polygram shown in red.*

Each of the images in Figure 8 is drawn in a CW direction (as can be seen using *FCLD* mode with $DL = L$ in [4]) and each has 60 subimages, but it is worth considering how that happens in each instance. The 5th line ends at the first subdivision past the top on the VF line from 0 to 5 in both panels 8(a) and 8(b). In 8(a), this is because 5 and 65 are MMI MOD 12·27 meaning that the endpoint of the 5th line is the first subdivision on the first line of the VF from vertex 0 to 5. In 8(b), this is because 5 and 115 are nMMI MOD 12·24 meaning that the endpoint of the 5th line in the image is the end of the 23rd or last subdivision (prior to the top of the 12,7-gram VF) on the last line of the VF from vertex 5 to 0. Both endpoints are in roughly the same location (1/27th or 1/24th from 0 along the VF line from 0 to 5), but 8(a) is 1 more than a multiple of $n·S$ and 8(b) is 1 less than a multiple of $n·S$. In both instances, the end of the first CG is at vertex 1. Figures 8(a) and 8(b) were also shown because, just like the typical clockface, there are 5 subimages (minutes) between each of the 12 (hours) vertices. Figure 8(c) is like 8(a) because 3 and 127 are MMI MOD 10·19.

MMI without a Clock

A particularly interesting non-clock-based MMI example from Chapter 8 of [2] is Figure 9(a), *Three Shape-Shifting Triangles*. To see this get drawn, choose *FCLD*, *FCFadingLD*, or *SLD* mode with $DL = L = 7$.

$J = 13$ $J = 7$
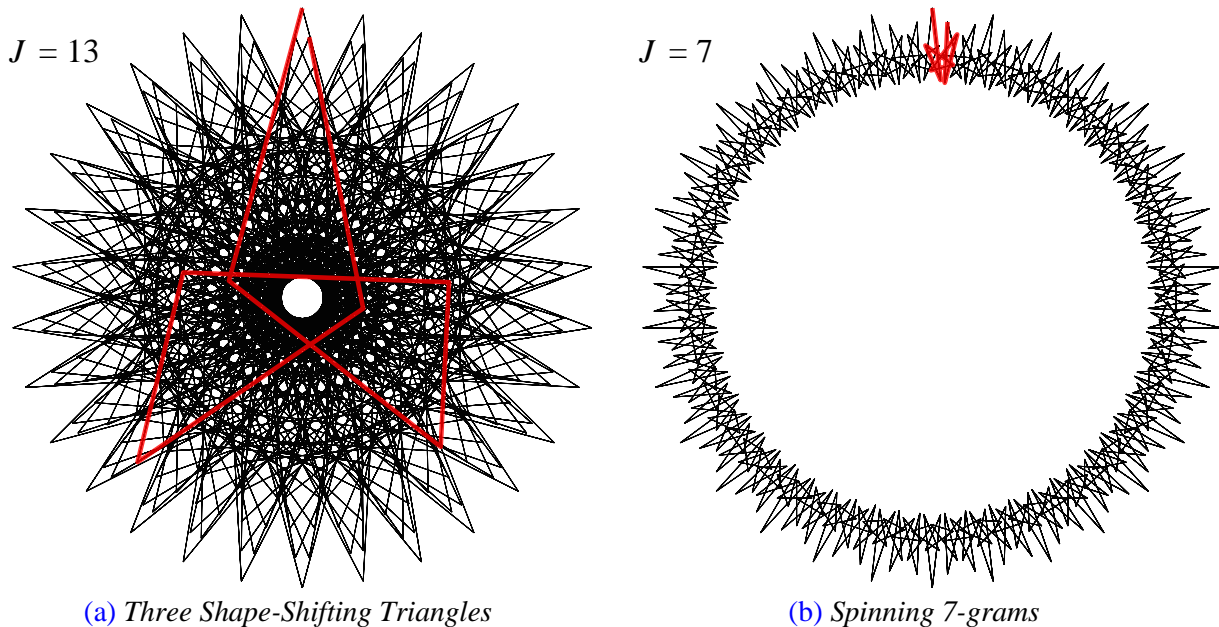
(a) *Three Shape-Shifting Triangles*   (b) *Spinning 7-grams*

Figure 9: *Two n = 30, S = 19, P = 163, 570-line images with first single-step of length 7 in red.*

Figure 9(a) is an 11-times around CCW-drawn, single-step of length 7 image that is completed in a bit more than 81 steps (since its first CG ends at vertex 19 = 13·163 MOD 30 and 19 > n/2; 11 = 30 – 19; 7·P = 1 MOD 570; and 81 = 567/7). This image, which is somewhat akin to an elaborate doily as a static image, is much more mesmerizing to watch get drawn. Since J is not part of the single-step calculation, altering J maintains the single-step of length 7 status even if VCF > 1. Among them, J = 1, 2, 5, 6, 7, 8, 9, 12 and 14 are all 7-point spinning stars. Figure 9(b) shows the J = 7 version, a CW-drawn 1-time around image.

Additional classes of *ESA* images based on MMI include stars as rotating polygons and stars (such as this (302, 1, 1, 121) pentagrams), porcupine polygons and stars (such as this (5, 25, 63, 1) pentagon, change to J = 2 for pentagram), and quivering polygons (such as this (8, 31, 63, 3) quivering quadrangles).

Summary and Conclusions

This paper introduces users to electronic string art on a regular polygon [2] in order to understand why some images appear that they were created like the second hand of a clock. To see an example that looks like a traditional second hand, click this (n, S, P, J) link (60, 2, 107, 23). These numbers are hyperlinkinked to this image, or click on [4] and adjust parameters accordingly, then choose *Drawing Mode, Fixed Count Line Drawing* with *Drawn Lines* = 2 and *Drawing Speed* = 1 to see a second hand in action. From here, change S (Subdivisions) to a larger value with *Drawn Lines* = S, and the image remains a clockwise one-time around 60-second image. Such images are created in 60 repetitive subimages. Each subimage ends one vertex clockwise farther around the 60-gon just like the ticking second hand on a clock. This one-step by one-step forward movement occurs because two of the parameters are multiplicative inverses modulo 60. Two numbers a and b are modular multiplicative inverses modulo c if their product has remainder 1 upon division by c, (this is denoted 1 = a·b MOD c; in *Excel* this is denoted MOD(a·b, c) = 1). Put another way, 60-second images provide a novel geometric interpretation of modular multiplicative inverses.

Acknowledgements

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