Embroidery-Hooping toward the Hyperbolic Trochoid

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Abstract

Consider a tortoise and hare ambling along a circular track holding the ends of a uniformly stretchable bungee cord between them. When viewing the bungee cord in time and space, the cords form an envelope of a curve. Should the relative speeds of the tortoise and hare remain constant, these curves are the well-known epicycloids and hypocycloids of Euclidean geometry. But what happens in hyperbolic geometry within the unit disk? The result is like visiting a county fair's hall of curved mirrors. We generate the curves both mathematically and with embroidery hoops, yarn, and weed-whacking string.

Introduction

Imagine an ideal, uniformly stretchable bungee cord whose ends are held by two runners, a tortoise T and a hare H, about a circular track C. H proceeds at speed rate p, and T at rate q. Now imagine a stroboscopic time-lapse, overhead photo of the bungee cord at those illuminated moments with respect to the track. For both Euclidean and hyperbolic geometry, these families of lines define a curve, called an *envelope*. What curves are they and how can we make them?

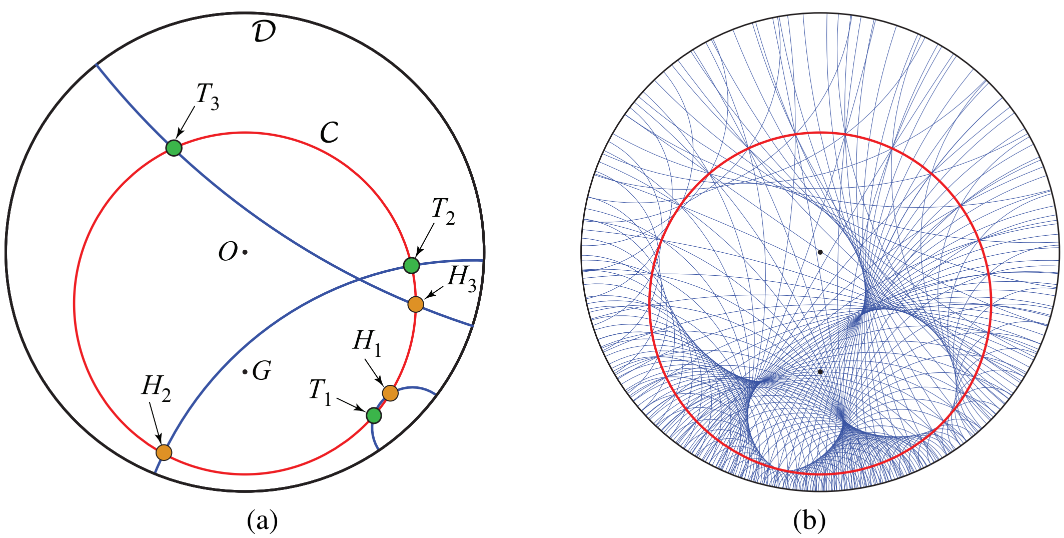
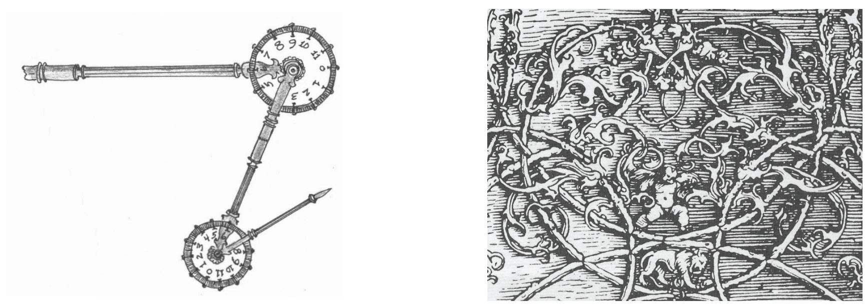


Figure 1: *Clover-leaf-like families of n lines where (a) $n = 3$ and (b) $n = 200$.*

For example, Figure 1(a) shows H and T moving counterclockwise about a circular track C within the unit disk $\mathcal{D}$ holding a bungee cord between them when $p = 4$ and $q = 1$. The tortoise and hare positions at three equally-spaced time intervals are indicated by points T_1 and H_1, and so on. Notice that the cords between them are circular arcs which, when extended, meet $\mathcal{D}$'s boundary at right angles. Points O and G are, respectively, $\mathcal{D}$'s center and C's center—all in accordance with the properties of hyperbolic lines and circles. Figure 1(b) shows a family of 200 hyperbolic lines while T completes a single lap about C, so creating what resembles a non-symmetric three-leaf clover. In Euclidean space, analogous envelopes are members of the trochoid curve family; see, for example, [2], [6], [7, chapter 10].

Much of the material in this presentation was developed for my college geometry, math-modeling, and, yes, even complex analysis classes over many years while striving to quicken the students' intuitive—and tactile—understanding of isometric motions (translations, rotations, reflections, and glide reflections) in both Euclidean and hyperbolic geometry. Among all the various families of simple planar curves, my vote for the most beautiful are the trochoids. Witness the perennial popularity of the *Spirograph* toy over the generations; a version named the *Speiragraph* was marketed as early as 1827, [8]. And the history of the trochoid family includes Albrecht Dürer and his self-modified compasses, such as the image in Figure 2(a); Figure 2(b) is Dürer's rendition of what is now known as the *limaçon*, being a snippet from his 1505 woodcut *The Circumcision*. And Copernicus's 1543 intricate heliocentric model borrows the notion of epicyclic trochoids from Ptolemy's solar system model. Incidentally, the limacon curve appears as a tortoise-and-hare bungee cord envelope when their speed ratio $\frac{p}{q}$ is two-to-one!



(a) (b)

Figure 2: *(a) Dürer's modified compass, author sketch, after [4, Bk I, figure 43]; (b) Dürer's limaçon, [3].*

To explore this tortoise-and-hare bungee cord puzzle we outline pertinent hyperbolic geometry elements; give easy-to-use *Möbius* transformation characterizations for hyperbolic isometries; illustrate them; sketch an algorithm for displaying the lines; and offer tips for constructing trochoids using embroidery hoops.

Finding a Hyperbolic Line's Center, Radius, and Ideal Points

Consider Figure 3(a) where A and B are points inside disk $\mathcal{D}$, points called *regular* points. The hyperbolic line $\mathcal{L}$ through A and B perpendicular to $\mathcal{D}$'s boundary is the arc of a Euclidean circle $\mathcal{E}$ with radius r and center $E = (a,\ b) \equiv a + ib$, $a,\ b \in \mathbb{R}$, where the x-y plane is identified with the complex plane $\mathbb{C}$ for ease of algebraic manipulation. Line $\mathcal{L}$'s endpoints P and Q on $\mathcal{D}$'s boundary are called *ideal* points. Given A and B, how may we find $\mathcal{L}$'s ideal points, center, and radius?

Here's a simple, albeit brute-force, way to find r and the two coordinates each for P and E. We exploit six independent relations: (i) the perpendicular bisector of $\mathcal{E}$'s chord AB passes through $\mathcal{E}$'s center E, (ii) the distance from E to A is r, denoted $|EA| = r$, (iii) $|EB| = r$, (iv) $|EP| = r$, (v) $|OP| = 1$, and (vi) Euclidean segments OP and EP meet at a right angle. Solving this nonlinear system on a computer algebra system gives all five solutions, including a dual solution pair for P—one of which gives Q's coordinates. In the special case where O lies on $\mathcal{L}$, then $\mathcal{L}$ is a diameter of $\mathcal{D}$ and corresponds to an arc of infinite radius.

Reflection, Rotation, and Translation within $\mathcal{D}$

Since this presentation is all about viewing a family of enveloping lines within the hyperbolic disk $\mathcal{D}$ as the family of lines is moved *isometrically*—that is, motion that leaves intact an object's inherent size and shape with respect to its topological universe—we'd like easy-to-use algebraic tools to do so. Here's the first of three tools. Note: The *conjugate* of $z = x + iy$ is $\overline{z} = x - iy$, where $x,\ y \in \mathbb{R}$ and $z \in \mathbb{C}$.

A Hyperbolic Reflection Algorithm

Let $\mathcal{L}$ be a hyperbolic line not containing the origin with Euclidean center $E \in \mathbb{C}$ and radius $r > 0$, then the composition of functions $F(z) = f^{-1} \circ g \circ f(z)$ is a hyperbolic reflection isometry, where

$$f(z) = \frac{z - E}{r}, \quad g(z) = \frac{1}{\overline{z}}, \quad \text{and} \quad F(z) = \frac{E\overline{z} - 1}{\overline{z} - \overline{E}}. \tag{1}$$

The functions in (1) may also be written equivalently in *Cartesian coordinates*. For example, $g(z)$ is equivalent to any of the following expressions:

$$g(z) = g(x + iy) = \frac{1}{x - iy} = \frac{1}{\overline{z}} = \frac{z}{\overline{z}z} = \frac{z}{|z|^2} = \frac{x + iy}{x^2 + y^2}. \tag{2}$$

Figure 3 is a proof without words that F reflects points about $\mathcal{L}$ so that region $\mathcal{R}$ of (a) is mapped to region

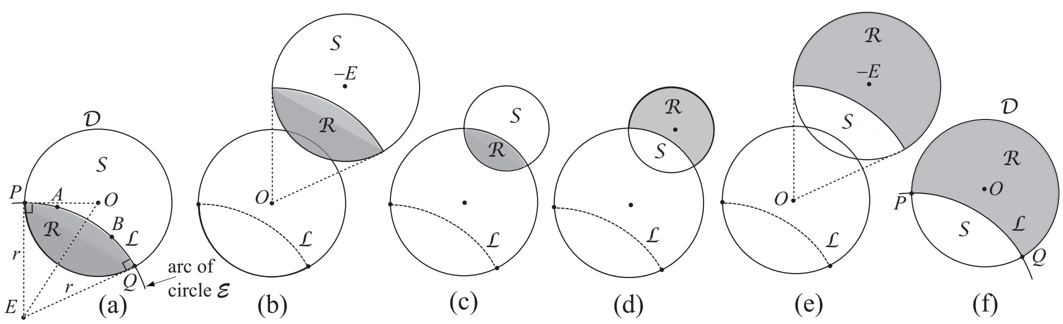


Figure 3: *Reflection isometry of $\mathcal{D}$ to $\mathcal{D}$: (a) region $\mathcal{R}$, (b) translation by $-E$, (c) scaling by $\frac{1}{r}$, (d) scaling each z by $\frac{1}{|z|^2}$, (e) scaling by r, (f) translation by E.*

$\mathcal{R}$ of (f). To insert a few words, imagine that $r > 1$. From (a) to (b), f translates region $\mathcal{R}$ by $-E$; from (b) to (c), region $\mathcal{R}$ shrinks when multiplied by $\frac{1}{r}$; from (c) to (d), each point z in $\mathcal{R}$ is scaled by $\frac{1}{|z|^2}$ as given by Equation (2); from (d) to (e), $\mathcal{R}$ is magnified by a factor of r; from (e) to (f), $\mathcal{R}$ is translated by E.

To sketch a formal derivation of (1), the vital key identity is $r^2 + 1 = E\overline{E}$ as given by the Pythagorean theorem, since $\triangle EPO$ is a right triangle. This identity also implies that $\overline{F}$ is a *Möbius transformation*, a function of the form $h(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha\delta - \beta\gamma \neq 0$; simply let $\alpha = \overline{E}$, $-\beta = 1 = \gamma$, $\delta = -E$. As shown in [1, Theorem 3.5.1, pp. 40–41], Möbius functions preserve angles. And since the conjugation transformation preserves angles, then F preserves angles. To show that F preserves hyperbolic distances, see standard references, such as [1].

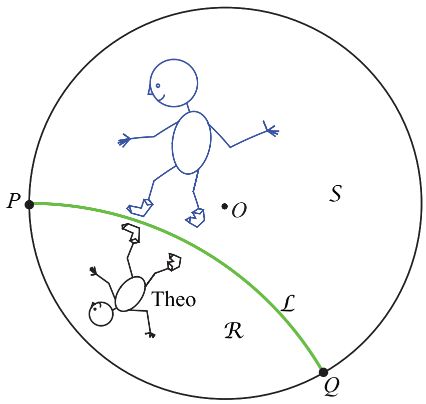
A Hyperbolic Rotation Algorithm

Let lines $\mathcal{L}$ and $\mathcal{M}$ intersect at point X in $\mathcal{D}$ where θ is the angle between $\mathcal{L}$ and $\mathcal{M}$, $0 \leq \theta \leq \frac{\pi}{2}$, as in Figure 4(b). Let F and G be reflection transformations about $\mathcal{L}$ and $\mathcal{M}$, respectively. The composition $G \circ F(z)$, being a reflection about $\mathcal{L}$ followed by a reflection about $\mathcal{M}$, is a rotation about X by angle 2θ—a phenomenon also true in Euclidean spaces. This rotation isometry $G \circ F(z)$ simplifies as the Möbius transformation with

$$\alpha = 1 + \overline{E_1}E_2 = \overline{\delta} \quad \text{and} \quad \beta = E_2 - E_1 = \overline{\gamma}. \tag{3}$$

Before introducing our third isometry, let's experiment with the first two. Rather than transforming a line to a line, we transform a stick figure named *Theo*—since discerning to where hands, feet, and head get mapped is readily apparent. Consider Theo in Figure 4(a) where $\mathcal{L}$ is the line with ideal points $P = (-1, 0)$ and $Q = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. $\mathcal{L}$ has radius $r = \sqrt{3}$ and center $T = (-1, -\sqrt{3})$, a third quadrant point outside of $\mathcal{D}$. By

Equation (1), a reflection about $\mathcal{L}$ maps region $\mathcal{R}$ to region $\mathcal{S}$, and thus Theo is reflected from below $\mathcal{L}$ to above $\mathcal{L}$. Note that Theo's boots on both sides of $\mathcal{L}$ are about the same size because they are near each other, whereas Theo's reflected oval body in region $\mathcal{S}$ appears larger than Theo's body in region $\mathcal{R}$—because objects appear to grow in size as they approach the origin in this hyperbolic universe.



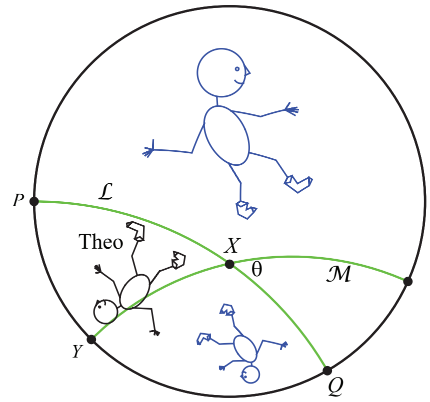


Figure 4: *Theo (a) reflected about $\mathcal{L}$, and (b) rotated about $X = (0, \sqrt{2} - \sqrt{3})$.*

In Figure 4(b), $\mathcal{M}$ is the line through ideal point $Y = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and regular point $X = (0, \sqrt{2} - \sqrt{3})$, which is also on $\mathcal{L}$. $\mathcal{M}$'s center is $U = (\sqrt{3} - \sqrt{2}, -\sqrt{3})$ and has radius $s = \sqrt{7 - 2\sqrt{6}}$. The acute angle θ between $\mathcal{L}$ and $\mathcal{M}$ is $\theta = \cos^{-1} \frac{(X-T)\cdot(X-U)}{rs} \approx 47.93°$. Reflecting Theo about $\mathcal{L}$ and then $\mathcal{M}$ is a counterclockwise rotation of 2θ about X, whereas reflecting about $\mathcal{M}$ and then $\mathcal{L}$ is a clockwise rotation of 2θ.

Now, what about translations? One might intuitively think that if lines $\mathcal{L}$ and $\mathcal{N}$ intersect at an ideal point, then a reflection about $\mathcal{L}$ followed by a reflection about $\mathcal{N}$ may be a translation, since the angle of rotation is zero. Consider Figure 5(a) where $A = (-\frac{1}{2}, -\frac{1}{2})$ is at Theo's heart; the reflection of A about $\mathcal{L}$ is $B = (-16 + 6\sqrt{3},\ 15 - \sqrt{3})/37$. Let $\mathcal{N}$ be the line between B and Q. Reflecting Theo about $\mathcal{L}$ followed by a reflection about $\mathcal{N}$ yields the image in the figure. In this figure, Theo is juggling a ball in his right hand, currently at point C, whose image when rotated 0 radians about ideal point Q is at point D. However if we also place a ball in Theo's left hand, the hyperbolic lines between each ball and its image under the combined reflections would each cross the conjectured *axis of translation* AB. So our intuitive idea fails. That is, rotation of an object about an ideal point appears to be a nontrivial rotation even though the angle of rotation is nought!

To find a translation algorithm much like (1) and (3), we exploit the theorem that every hyperbolic isometry is a Möbius transformation or the conjugate of one; see [1, Theorem 7.4.1, pp. 137–138].

A Hyperbolic Translation Algorithm

If we agree that a translation should be an isometry mapping point A to B so that an object at A glides along the the *axis of translation* AB whose ideal points are P and Q, we must find a Möbius transformation $h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ with fixed points P and Q and $h(A) = B$. Setting δ to unity and solving this system of three equations for α, β, γ, and then scaling all four parameters appropriately yields a unique Möbius transformation (modulo any nonzero scaling of this parameter system):

$$\alpha = PQ - B(P + Q - A),\ \beta = PQ(B - A),\ \gamma = A - B,\ \text{and } \delta = PQ - A(P + Q - B). \tag{4}$$

To translate Theo's heart from A to B in Figure 5(b), let $\mathcal{W} = AB$ be the axis of translation. $\mathcal{W}$'s ideal points are $S \approx -0.686 - 0.728i$ and $T \approx -0.149 + 0.989i$; via (4), $\alpha \approx 1.044 - 0.335i$, $\beta \approx 0.776 + 0.507i$,

$\gamma \approx -0.348 - 0.859i$, and $\delta \approx 0.529 - 0.960i$. With these parameters, the translation arcs under h from each hand to its image are parallel (in the sense that the lines fail to intersect) to $\mathcal{W}$.

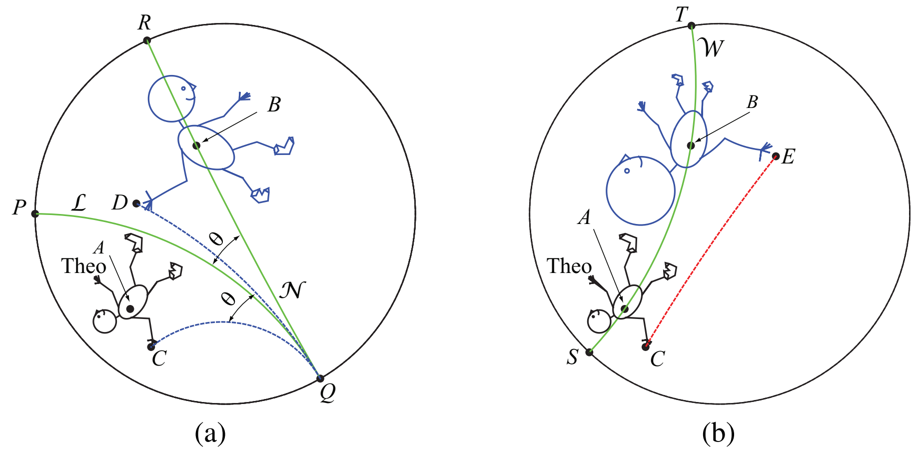


Figure 5: *Theo (a) rotated about ideal point Q, and (b) translated along line $\mathcal{W}$.*

A Display Algorithm for Hyperbolic Lines

In this section, we construct an envelope step-by-step.

Step I

Choose a track radius R with center O. In particular, let's follow Figure 1 with $R = \frac{4}{5}$ and $\frac{p}{q} = \frac{4}{1}$. Suppose that, after H and T start at the same track point, the overhead strobe-light flashes $n = 200$ times until T completes a track lap. By way of Euler's well-known identity, the photos record T's locations at

$$t_j = \frac{4}{5} e^{\left(\frac{2\pi i j}{n}\right)} = \frac{4}{5}\left(\cos\frac{2\pi i j}{n} + i\sin\frac{2\pi i j}{n}\right),$$

for each j, $0 \leq j \leq n$. Meanwhile, since H travels four times as quickly, H's locations are at $h_j = \frac{4}{5}e^{\frac{8\pi i j}{n}}$. For example, the first three paired locations (h_j, t_j) on this track are

$$\Big\{(0.8,\ 0),\ (0.8, 0)\Big\},\quad \Big\{(0.79,\ 0.1),\ (0.7996,\ 0.025)\Big\},\quad \Big\{(0.77,\ 0.2),\ (0.7984, 0.05)\Big\}, \ldots.$$

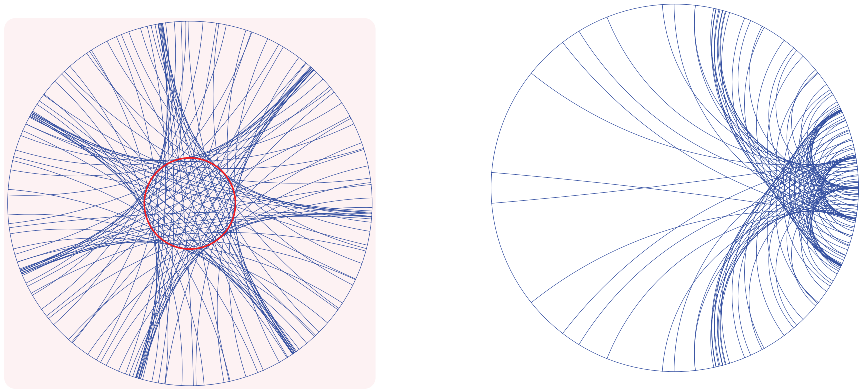
Step II

Choose an isometry k. To follow Figure 1, let k map $A = O$ to $B = G = -\frac{i}{2}$, the respective centers of hyperbolic circles $\mathcal{D}$ and C: $k(z) = \frac{z - \frac{i}{2}}{\frac{iz}{2} + 1} = \frac{2z - i}{iz + 2}$, via Equation (4), where $P = i = -Q$. For each j, calculate $\{k(h_j),\ k(t_j)\}$, the paired locations for H and T along this new track, obtaining the sequence

$$\{(0.56,\ -0.66), (0.53, -0.69)\},\ \{(0.61,\ -0.59), (0.54, -0.68)\},\ \{(0.64, -0.52),\ (0.55, -0.67)\}\ldots.$$

Step III

For each $\{k(h_j),\ k(t_j)\}$, where $k(h_j) \neq k(t_j)$, determine the hyperbolic line through that ordered pair. That is, in Figure 1(a), $T_1 = k(t_2)$, $T_2 = f(t_{29})$, and $T_3 = k(t_{56})$, where 2, 29, 56, ... is an arithmetic sequence. Plotting all the hyperbolic lines yields Figure 1(b), $1 \leq j \leq 200$.



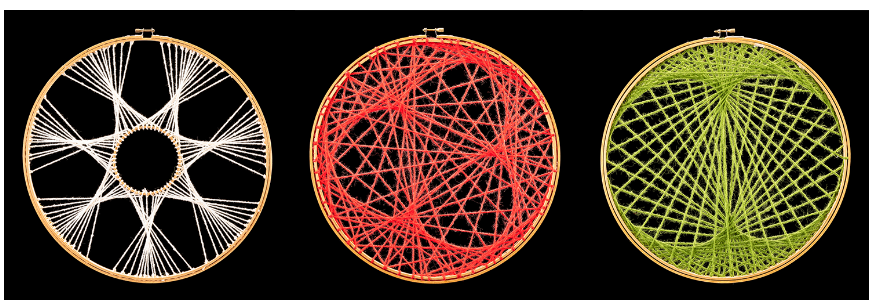
(a) (b)

Figure 6: *A starfish-like trochoid of $n = 100$ hyperbolic lines, $p = 5$, $q = -2$: (a) track center at O, radius $\frac{1}{4}$, (b) track center at $(\frac{7}{10}, 0) \equiv \frac{7}{10}$.*

For a second example, let $n = 100$, $p = 5$, $q = -2$, and $R = \frac{1}{4}$, so that H and T proceed in opposite directions about the track. Figure 6(a), shows the family of lines when H and T proceed about the track centered at O, and Figure 6(b) shows the family when they proceed about the track after it has been translated by $\frac{7}{10}$. Notice that the enveloped curves appear outside the tracks, rather than inside.

A Hyperbolic Hoop Gallery in Yarn, Zip Ties, and Weed-Whacking String

After visiting a local craft shop for a supply of embroidery hoops, various skeins of yarn or twine, colored zip ties (cable ties), and colored weed-whacking string, let's construct physical models. The total material expense for all models in Figures 7 through 10 was less than \$30 US.

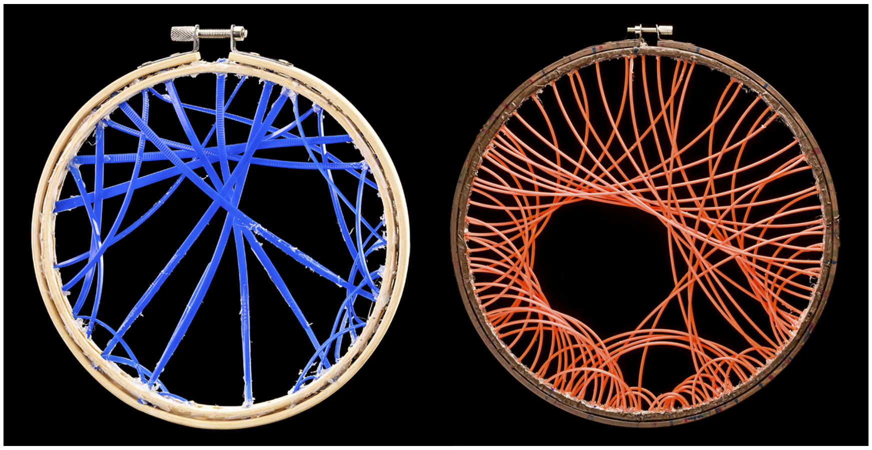


(a) (b) (c)

Figure 7: *Trochoid families dressed in yarn: (a) $(p, q) = (7, -1)$, (b) $(p, q) = (5, 2)$, (c) $(p, q) = (3, 1)$.*

The Figure 7 trochoids are made with yarn in hoops of 14-inch diameters; all track radii are $R = \frac{1}{4}$ with center O. Since these tracks are near O, hyperbolic lines within the tracks appear to be nearly straight. In (a), two hoops are used, with the track itself being a 4-inch hoop; lines within this track are omitted. Figure 8 trochoids are made with zip ties and weed-whacking string, respectively, with $R = \frac{4}{5}$, and with respective isometries being (a) the identity and (b) the Möbius function $\frac{2z-i}{iz+2}$. In contrast to Figure 7, the lines resemble circular arcs. Since both zip ties and weed-whacking string have natural curvature, that trait persists after adhering tie-ends or string-ends to the hoops. Bits of errant hot glue on some of the strands of (a) create an illusion of frost on the trochoid. In (a), the hoop is the track; and in (b) the hoop is the boundary of

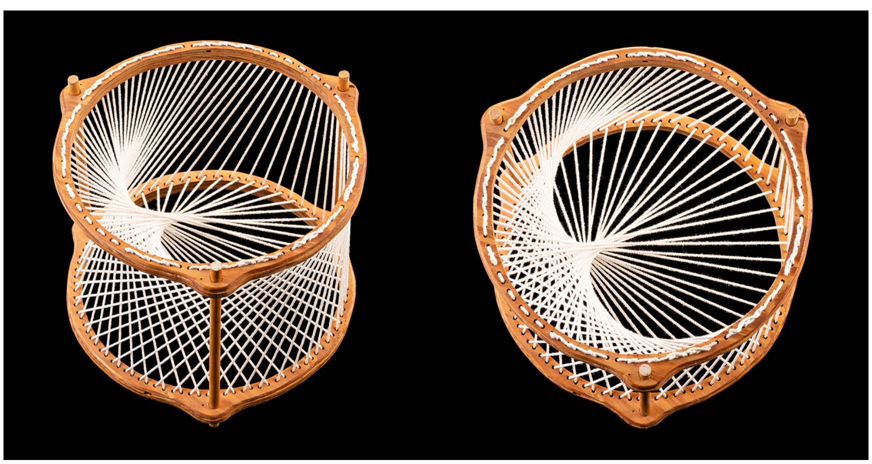
$\mathcal{D}$. Furthermore, (b) is a rendition of Figure 1(b), albeit some strands were omitted because of strand-end overcrowding near the lower portion of the hoop.



(a) (b)

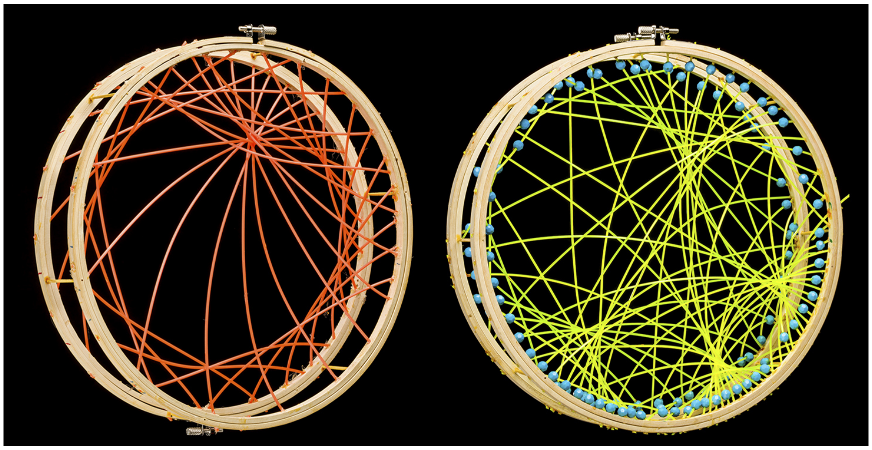
Figure 8: *Trochoid lines families attired in zip-ties and weed-whacking string: (a)* $(p,\ q) = (2,\ 1)$ *in a 6-inch hoop, (b)* $(p,\ q) = (4,\ 1)$ *in a 10.5-inch hoop.*

To address this challenge of threading overly-many strands through a short arc region along a hoop, why not use a hare hoop and a tortoise hoop? Figure 9 shows the result; upon switching our view from side to overhead, Dürer's limaçon of Figure 2(b) appears, where $\frac{p}{q} = \frac{2}{1}$. Figure 10(a) re-renders Figures 8(a) and 9 using twin hoops separated by wooden toothpicks with 35 strands, $R = \frac{1}{4}$, and a translation isometry mapping O to 0.95. For (b), $\frac{p}{q} = \frac{5}{1}$, with 81 strands of lesser gauge than used in (a); $R = \frac{1}{2}$; the chosen isometry is $\frac{(1+i)\overline{z}-1}{\overline{z}-(1-i)}$ from Equation (4); the plastic beads on the strands near the hoops' rims signify the tortoise and hare positions, respectively, along the tracks of the upper and lower hoops.



(a) (b)

Figure 9: *The limaçon reappears! A 3-D trochoid in 13-inch plywood hoops and 12-inch long* $\frac{1}{4}$*-inch dowels: (a) side view, and (b) overhead view.*



(a) (b)

Figure 10: *3-D trochoids on 14-inch hoops separated by toothpicks: (a)* $(p, q) = (2, 1)$ *with* $n = 35$ *strands, and (b)* $(p, q) = (5, 1)$ *with* $n = 81$ *strands.*

Summary and Conclusions

As we have seen, this tortoise and hare bungee cord puzzle involves forming families of hyperbolic lines that envelope trochoidal-shaped curves. For algorithms to recover the curve envelope itself, see [5]. For further explorations, the reader might experiment with what happens when p and q assume non-integer values.

Acknowledgements

Thanks to Professor Joe Strickland of King University's Digital Media Art & Design Department for photographing my trochoid models for Figures 7 through 10.

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