What Can We Learn from Non-Euclidean Paper?

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Abstract

The author introduces a technique using 3D printing and vacuum forming for producing paper-like plastic sheets endowed with non-Euclidean geometry. He then explores a number of potential applications of this non-Euclidean "paper" to education and illustration of concepts from geometry including Gaussian curvature, surface isometries, constructability in non-Euclidean geometries, and more.

Introduction

In 2012, Alperin, Hayes, and Lang presented a two-headed origami crane folded from a sheet of "hyperbolic paper"—paper which had been chemically softened and allowed to conform to the surface of a 3D-printed mold with constant negative curvature [1]. Eleven years later, we revisit the idea of non-Euclidean paper with an eye towards education and illustration, asking ourselves the titular question, "what (else) can we learn from non-Euclidean paper?"

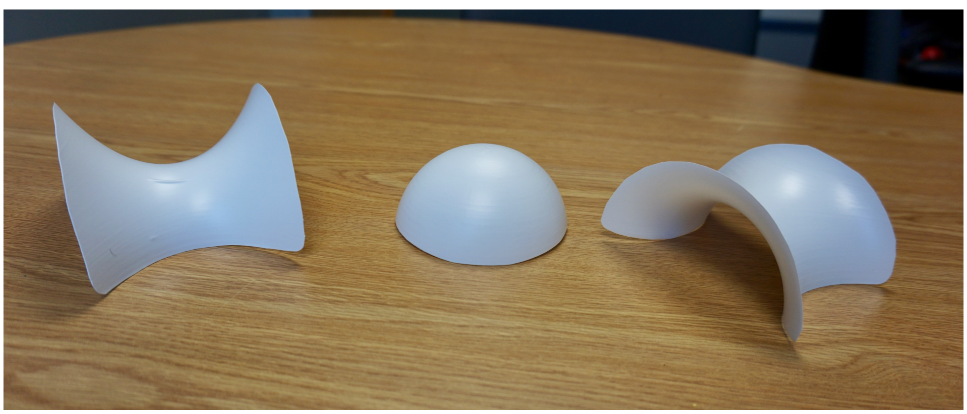


Figure 1: *Sheets of non-Euclidean paper.*

After some background on the mathematics behind non-Euclidean paper, we describe a different production technique for making non-Euclidean paper—or rather, paper-like sheets of plastic—using vacuum forming, the products of which are shown in Figure 1. For many would-be makers and users of non-Euclidean paper, this may be more accessible or economical than the technique introduced by Alperin, Hayes, and Lang (although certain shortcomings are encountered). We then present several novel ways in which non-Euclidean paper can be used to illustrate geometric concepts to learners through interaction with the material. These include a sampling of ideas of what one might draw on non-Euclidean paper, along with physical demonstrations of Gauss's *Theorema Egregium*, surface isometries, and the nonexistence of an isometric immersion of the hyperbolic plane.

Ultimately, we hope that this article will serve as an invitation to the reader, both to imagine new demonstrations and uses for non-Euclidean paper and to ask new questions of the material.

Mathematical Background

Paper and Intrinsic Geometry

The paper we encounter in our daily lives has several physical properties that can be modeled mathematically. First, we typically think of paper as being two-dimensional—its thickness being small enough to ignore—and so a piece of paper positioned in our three-dimensional world can be thought of as a surface.

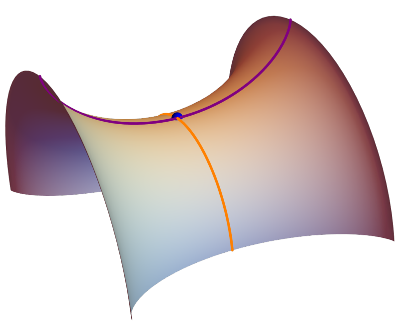
Paper easily bends and folds, but it does not stretch or contract. Importantly, the shortest distance along a sheet of paper between a pair of points remains fixed under bending or folding. Angles and areas remain similarly unchanged. In mathematical vocabulary, we say that any two configurations of a sheet of paper are *isometric* to each other, and that they have the same *intrinsic* geometry.

The final feature of traditional paper that we will focus on is that it can lay flat; it has the same intrinsic geometry as a piece of the Euclidean plane, and so all of Euclid's axioms of geometry apply to lines, circles, etc. drawn on the surface. Henceforth, we refer to traditional paper as *Euclidean* paper. Euclidean paper isn't always flat in the usual sense of the word since it can be bent or folded, changing its *extrinsic* geometry, but since distances between points are preserved, the intrinsic geometry is preserved.

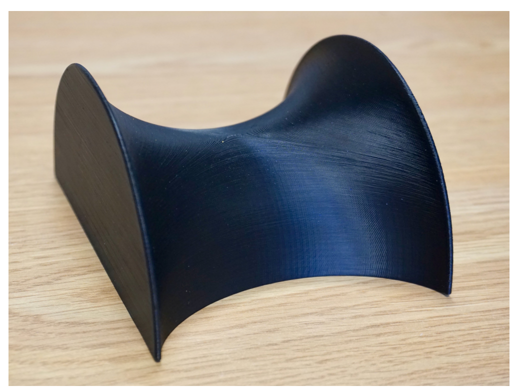
Paper that cannot lie flat—which is not isometric to a piece of the Euclidean plane—will be called *non-Euclidean* paper. A piece of non-Euclidean paper still has the property that its intrinsic geometry is preserved under bending and folding since distances between points along the surface do not change; that is, any two surfaces formed from the same sheet will be isometric to one another.

Gaussian Curvature and the Theorema Egregium

At a point P on a locally oriented smooth surface, one can pick a tangent vector to the surface and draw the geodesic on the surface that starts in that direction. The signed curvature (positive if curving in the direction of the normal vector, negative otherwise) of that geodesic can be called a *directional curvature* at the point P. The maximum and minimum directional curvatures at P are called the *principal* curvatures, k_1 and k_2 respectively, and their product $K = k_1 k_2$ is called the *Gaussian curvature* (or just *curvature*) of the surface at P, as illustrated in Figure 2(a).



(a)



(b)

Figure 2: *A surface with constant negative Gaussian curvature. (a) We can observe that the surface has negative curvature at the marked point as follows. The purple curve is the geodesic through the point with greatest upward curvature, and the orange curve is the geodesic with the greatest downward curvature, so their signs are considered to be opposite, and therefore their product K is negative. (b) A 3D-printed model of the same surface, which will be used for thermoforming hyperbolic paper. Note that the edges are rounded to prevent tearing.*

The directions in which these principal curvatures occur are called the *principal directions*, and they are necessarily perpendicular to each other. A point with positive curvature looks locally like a sphere—curving towards the same normal direction along all geodesics through the point—while a point with negative curvature looks locally like a saddle or potato chip—with some geodesics curving in one normal direction and others curving in the opposite normal direction.

While the calculation of K relies on the extrinsic properties of the surface (how it is positioned in the ambient space), Gauss's *Theorema Egregium* states that it is actually an intrinsic property and is preserved under isometry [5]. This means that a marked point on non-Euclidean paper will have the same Gaussian curvature regardless of how that paper is (smoothly) bent in space.

Surfaces of Constant Gaussian Curvature

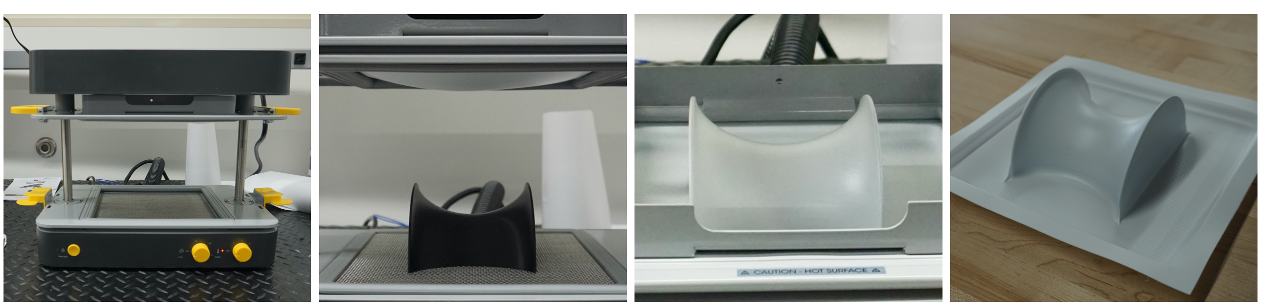
The production technique described in this article will allow us to produce paper with a wide range of possible intrinsic geometries, but surfaces with *constant* Gaussian curvature constitute an important set of examples.

Applying the definition of Gaussian curvature to the Euclidean plane, we see that every point on the plane—and therefore on any surface formed from Euclidean paper—has zero curvature. A sphere, on the other hand, has constant *positive* curvature, so any surface formed from a sheet of *spherical paper* (paper whose initial shape is part of a sphere) will also have constant positive curvature.

A general classification of surfaces of revolution with constant Gaussian curvature can be found, for example, in [4] (Section 3.3, Exercise 7). In particular, there exist surfaces of revolution with constant *negative* curvature, such as the one pictured in Figure 2. By forming over such a surface, we can obtain non-Euclidean paper with constant negative curvature, which we'll call *hyperbolic paper* since it can serve as a model for a piece of the hyperbolic plane in the theory of non-Euclidean geometry [3].

Production Technique

Thermoforming is a process by which a heated plastic sheet is formed onto a mold called a *buck*. When the forming is achieved using a vacuum pump, the process is called *vacuum forming*. Thermoforming in general is widely used in manufacturing (think of plastic packaging for toys), but has also been applied to creating mathematics instructional tools for multivariable calculus [2, 9]. In the author's application of the process, one starts by 3D-printing the buck using ABS plastic filament (which has a higher melting point than other commonly used 3D-printing materials like PLA) such as the one shown in Figure 2(b).



(a) (b) (c) (d)

Figure 3: *Photos of the thermoforming process: (a) Mayku Formbox, (b) a heated sheet of HIPS can be seen sagging below the heating element above the buck, (c) after pulling the heated sheet onto the buck, (d) the formed sheet before trimming.*

While vacuum formers range widely in size, to create the non-Euclidean paper presented here, the author used a desktop Mayku Formbox, shown in Figure 3(a), which forms 23.5 cm square sheets of plastic. One fixes the plastic sheet to a frame that slides vertically along four pillars. At the top of the machine is a heater, so one starts by raising the frame close to the heater and placing the buck on the bed below. Once the plastic sheet has softened sufficiently (indicated by sagging), one simultaneously lowers the frame to drape the plastic over the buck and turns on the vacuum pump, which sucks through an array of tiny holes at the bottom. After cooling, the buck is then extracted from the formed plastic sheet. Each step is shown in Figure 3.

After experimenting with several types of plastic, the author had the most success with 0.5 mm thick high impact polystyrene or HIPS (think to-go coffee cup lids), and 0.25 mm thick PETG (think plastic soda bottles). A comparison of the resulting paper is outlined in Table 1.

Table 1: *Comparison of HIPS and PETG paper.*

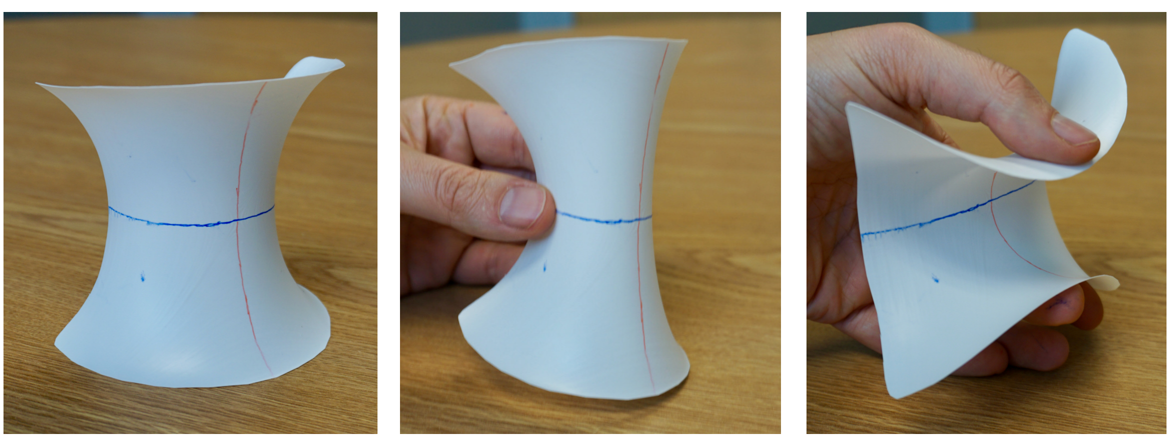
0.5 mm HIPS	0.25 mm PETG
opaque, white	transparent
stiffer	more pliable
markable with permanent markers	markable with permanent markers
	markable and erasable with dry-erase markers

Other types of plastic like acetate and polycarbonate can be made into very thin sheets that behave more like paper than HIPS or PETG, but they have material properties that make thermoforming difficult. Since PETG is transparent, all of the non-Euclidean paper shown in the photographs in this article is made from HIPS.

While many libraries, maker spaces, and college campuses have 3D printers available for public use, vacuum formers are less commonly available (although the author was able to use one at his university's design lab). Desktop vacuum formers like the Mayku Formbox can be purchased for under $1000 (US) as of this writing (although a separate shop vacuum is needed). Various types of plastic can be purchased for about $1 (US) per sheet depending on the thickness and material.

Demonstrations

In this section we present several demonstrations of geometric ideas that are possible with non-Euclidean paper. These demonstrations were developed using the plastic paper produced using the vacuum forming techniques described above.



(a) (b) (c)

Figure 4: *Photos demonstrating Gauss's* Theorema Egregium. *The product of the curvatures of the two marked geodesics remains constant in each photo, so that when the magnitude of one geodesic's curvature is increased, the other decreases.*

Illustration of Gauss's **Theorema Egregium**

This demonstration works best with paper that is slightly rigid but which bends with a gentle squeeze, such as that made from HIPS plastic shown in the figures above. One starts with a piece of hyperbolic paper, and picking a point on the surface, one highlights the geodesics of maximal and minimal signed curvature as shown in Figure 4(a).

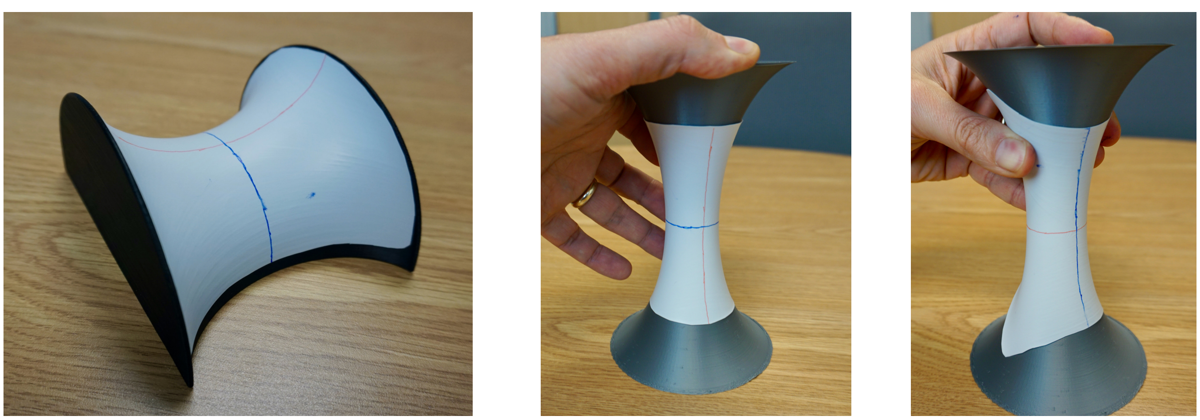
Pinching the ends one of the highlighted geodesics, one can increase the absolute value of one of the principal curvatures. However, since the Gaussian curvature—the product of the two principal curvatures—must stay constant, this will have the effect of decreasing the absolute value of the other principal curvature, resulting in a straightening of the other highlighted geodesic as shown in Figure 4(b). One can also pinch the other highlighted geodesic and observe the other one straighten as in Figure 4(c).

The same concept works on a piece of spherical paper. Starting with a hemisphere, one can draw two perpendicular geodesics through the center and perform the same demonstration.

Surface Isometries

As in the Mathematical Background section, we say two surfaces are *isometric* if they have the same intrinsic geometry. We can illustrate isometries between pairs of surfaces by showing that both surfaces can be formed by the same piece of paper. For example, one can show that a piece of the plane is locally isometric to the cylinder and to the cone, since Euclidean paper can be shaped into any of these three surfaces.

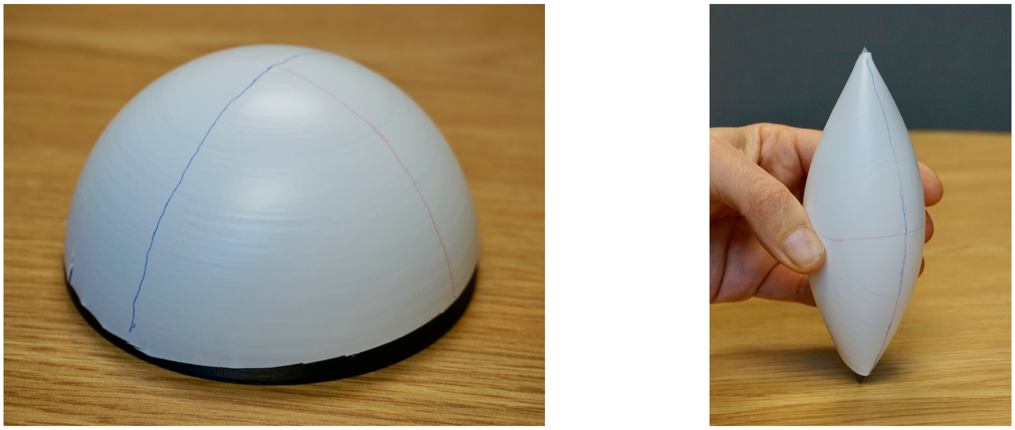
In general, a theorem of Minding states that any small patch of a surface with constant Gaussian curvature is isometric to a patch on any other surface with the same constant Gaussian curvature [8]. This can be illustrated by showing that a sheet of hyperbolic paper will lie flush against any surface with the same constant negative Gaussian curvature as shown in Figure 5. Also, since any patch of a surface with constant Gaussian curvature is isomorphic to any other patch, a constant curvature piece of paper can be slid along and rotated against a surface with the same curvature while laying flush with the surface as shown in Figures 5(b) and 5(c).



(a) (b) (c)

Figure 5: *A sheet of hyperbolic paper laying flush against different surfaces: (a) laying against the original buck, (b) laying against a different surface with the same constant negative Gaussian curvature, (c) laying against the same surface as in (b), but this time rotated* 90° *(notice that the signs of the curvatures are flipped).*

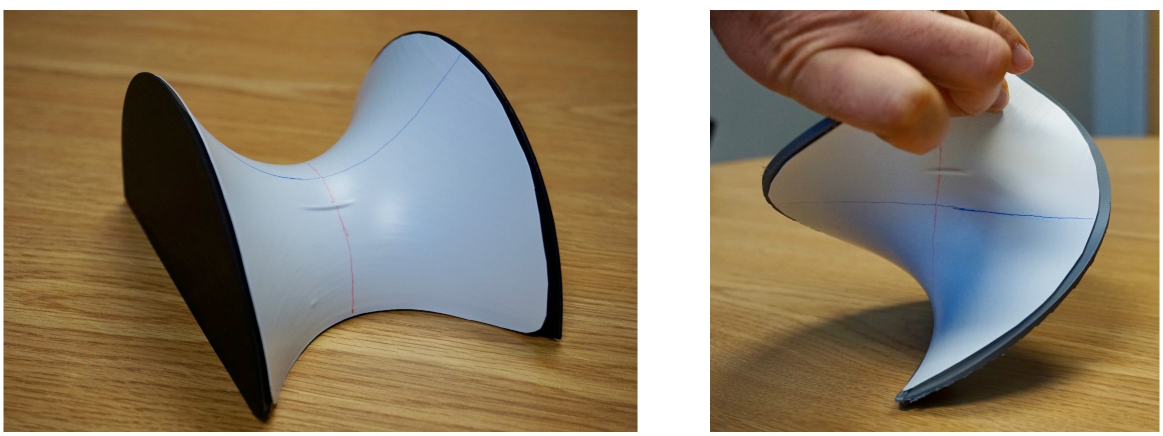
Somewhat surprisingly, there are surfaces other than the sphere with constant positive Gaussian curvature (albeit no other closed ones). For example, a piece of spherical paper will lie flat against the (American) football-shaped surface shown in Figure 6 away from the singular points.



(a) (b)

Figure 6: *The same sheet of spherical paper lying flush against two different surfaces of the same constant positive Gaussian curvature:* 6(a) *a sphere, and* 6(b) *an (American) football-shaped surface.*

As a final example, we illustrate a surface isometry that resonated deeply with the author when he first encountered these concepts. Patches of two seemingly dissimilar surfaces with non-constant Gaussian curvature, the catenoid and helicoid, can be shown mathematically to be isometric [4]. By taking a piece of non-Euclidean paper formed along the catenoid and aligning it just right along the helicoid, it can be made to lie flush as shown in Figure 7.



(a) (b)

Figure 7: *Demonstration showing that the catenoid (a) and helicoid (b) are locally isometric.*

Non-existence of a smooth hyperbolic plane in $\mathbb{R}^3$

It is a classic result of David Hilbert that there is no smooth isometric embedding of the entire hyperbolic plane into Euclidean three-space [6]. Although a patch of the hyperbolic plane can be smoothly isometrically embedded, for example, along a constant negative curvature surface like the one pictured in Figures 2(b), the singular edge of the surface represents the limit of how far the isometry can reach. One can demonstrate the failure beyond this limit by attempting to slide a piece of hyperbolic paper beyond the singular edge; the paper "pushes back," physically resisting the stretching, contracting, or introduction of nonsmooth creases that would need to take place to go further. If forced, the paper corrugates as shown in Figure 8.

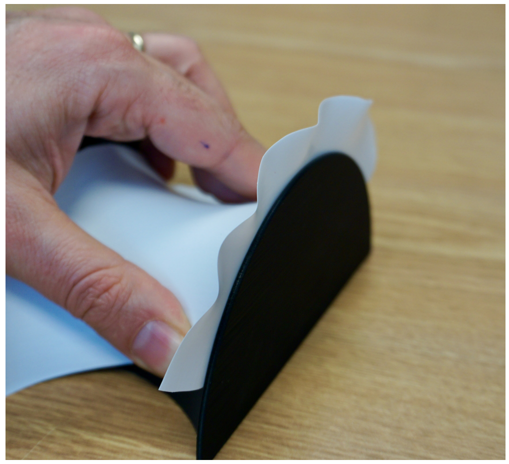


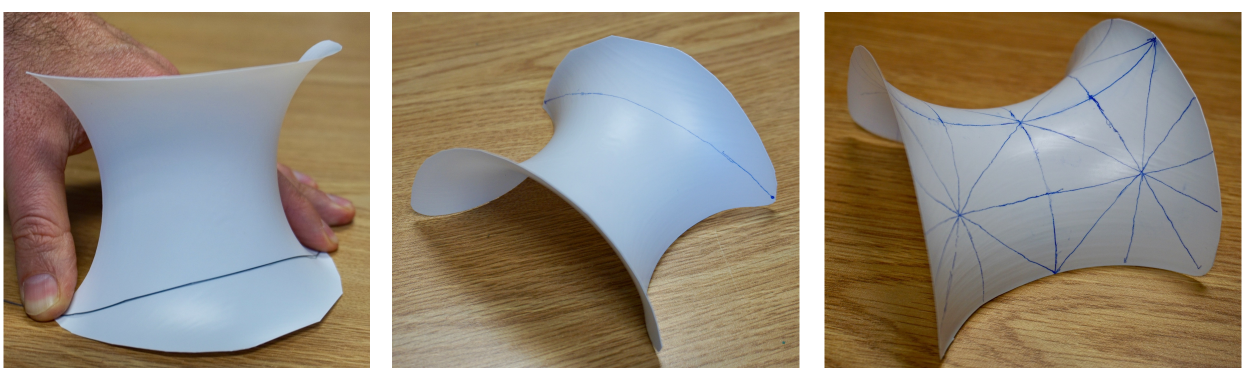
Figure 8: *Attempting to push a piece of hyperbolic paper past the singular curve.*

Drawing Curves

In order to illustrate geometric ideas, we want to be able to draw pictures. One can use permanent felt-tipped markers on either HIPS or PETG paper. Depending on the learning context, reuse of the paper may be desirable, in which case dry-erase markers can be used on PETG.

Rather than explaining in depth a single demonstration that can be achieved by sketching a curve on a sheet of non-Euclidean paper, we will instead include a short (non-comprehensive) list of concepts that could be illustrated on such paper. These concepts all deal with the intrinsic geometry of surfaces, which would be preserved under bending and folding of the paper.

Geodesics A *geodesic* on a surface is a locally length-minimizing path. Intuitively, it is an extension of a shortest path along a surface between a pair of points. A geodesic drawn on a surface modeled by non-Euclidean paper will remain a geodesic if the paper is bent or folded. By stretching a rubber band or taut string across the surface, one can trace a geodesic on a curved surface as shown in Figure 9(a), although this method only works if the geodesic curves away from the side of the paper being drawn on (else the string will lift off of the paper). Another method results from tracing along a thin ribbon laid flush along the surface since a ribbon with small but nonzero width will naturally follow geodesics [7].



(a) (b) (c)

Figure 9: *Geodesic tiling on hyperbolic paper: (a) stretching a string between two points to find the shortest path, (b) a single geodesic drawn by tracing the string, (c) several geodesics drawn to create a tiling of the hyperbolic plane by congruent triangles.*

Hyperbolic tilings If one can use non-overlapping congruent copies of a given shape (or set of shapes) to cover a surface, we call that a *tiling* of the surface. Just as the usual grid-shaped tiling of the Euclidean plane by squares helps us visualize its flatness, tilings of non-Euclidean paper can help us visualize its curvature. As an example, while triangles in the Euclidean plane have angles adding to 180°, Figure 9(c) shows a tiling by right triangles with angles 36°, 45°, and 90° (which add to 171°) on a piece of hyperbolic paper.

Non-Euclidean constructions. One can extend the idea of "ruler-and-compass" constructions in the Euclidean plane to non-Euclidean geometries. While a ruler corresponds to the ability to draw geodesics as described above, the author admits that he isn't sure how to easily replicate a compass (the ability to find the set of points equidistant from a given one along the surface).

Non-Euclidean Origami

The *raison d'être* of Alperin, Hayes, and Lang's original non-Euclidean paper, origami remains a topic of mathematical interest and is a wonderfully aesthetic demonstration of non-Euclidean geometry at work. Unfortunately, plastic paper is a poor substitute for cellulose paper when it comes to folding. While HIPS or PETG paper will hold a crease (yielding another possible method for drawing geodesics), they cannot be folded into anything so complicated as the hyperbolic crane.

Acknowledgements

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References

[1] R. C. Alperin, B. Hayes, and R. J. Lang. "Folding the Hyperbolic Crane." *The Mathematical Intelligencer*, vol. 34, 2012, pp. 38–49.

[2] J. Chen, K. Delp, and S. Paul. "Manipulative Calculus: Active Learning with 3D Models." *Proceedings of Symposia in Applied Mathematics*, to appear.

[3] H. S. M. Coxeter. *Introduction to geometry*, 2nd ed. Wiley, 1991.

[4] M. P. do Carmo. *Differential Geometry of Curves and Surfaces*, 2nd ed. Dover Publications, 2016.

[5] K. F. Gauss. *General Investigations of Curved Surfaces*, (ed. P. Pesic), Courier, 2005.

[6] D. Hilbert. "Über Flächen von Constanter Gaussscher Krümmung." *Transactions of the American Mathematical Society*, vol. 2, no. 1, 1901, pp. 87–99.

[7] D. Henderson. *Differential Geometry: A Geometric Introduction*, 1st ed. Prentice Hall, 1997.

[8] F. Minding. "Wie sich entscheiden lässt, ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht; nebst Bemerkungen über die Flächen von unveränderlichen Krummungsmaasse." *Journal für die reine und angewandte Mathematik*, vol. 2, pp. 370–387.

[9] A. Wangberg and B. Johnson. "Discovering Calculus on the Surface." *PRIMUS*, vol. 23, no. 7, 2013, pp. 627–639.