Interchangeable Origami Wallpaper Patterns

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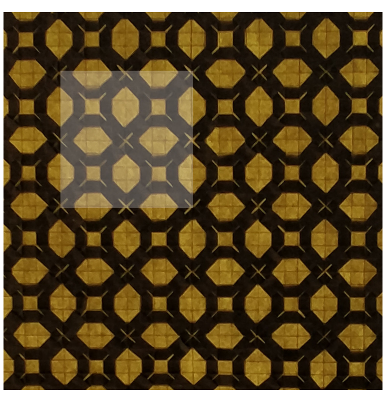
Abstract

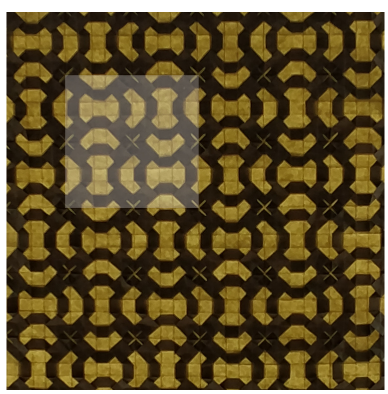
This article presents origami designs realizing the 17 wallpaper patterns, which are distinguished by the isomorphism types of the groups of plane isometries that preserve them. The origami models are adaptations of each other, physically realized through a selective reorientation of pleats throughout the pattern. All 17 are adapted from two basic patterns, one for the five patterns that involve 3-fold rotational symmetry, and another for the 12 patterns whose rotation orders are limited to 1, 2 or 4. It is proposed that origami provides an excellent environment for artistic and mathematical exploration of wallpaper patterns and the relationships among their symmetry groups. This is due to the fact that origami models may include local elements that are physically adjustable, and such adjustments may be selected to alter the symmetry group of the overall pattern.

Introduction

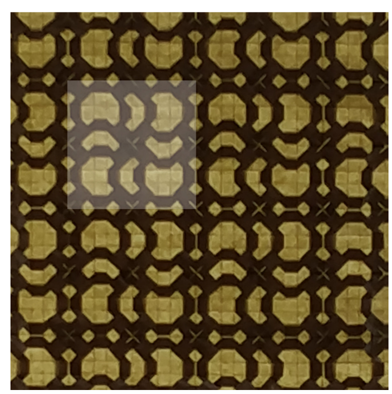
Figure 1 below shows three photographs of the same folded 50cm $\times$ 50cm sheet of 80g/m^3 Satogami paper, attached to a window and backlit by natural light. The photographs were taken a few minutes apart. What happened in these minutes was a reorientation of selected pleats, preserving some symmetries and eliminating or introducing others. The three pictures can be related spatially by noting the fine pale crosses or "$\times$" shapes in each, at the 25 interior points of a 6×6 square grid over the image. These features are identical in the three images, and untouched in the transformations between them. The three patterns shown all have 4-fold (90°) rotational symmetries, and realize the three distinct wallpaper groups possessing symmetries of this type, namely $p4m$, $p4g$ and $p4$.



(a) $p4m$



(b) $p4g$



(c) $p4$

Figure 1: *Backlit origami models for the wallpaper groups possessing 4-fold rotations.*

We propose that the craft of origami provides an extraordinarily rich physical environment for creative exploration of geometric and algebraic principles that describe periodic patterns in the plane. This is due to the tactile nature of the activity, and especially to the scope for local adjustment that is inherent to its folding operations, which facilitates transformations like those shown in Figure 1.

In the next section, we give some background on periodic patterns and their symmetries. In Section 3, we outline the reasoning that leads to the classification of patterns with 4-fold rotations into three types. In Section 4, we revisit the $p4m$ pattern of Figure 1(a), describe its construction, and elaborate on how to modify it to the pattern of Figure 1(b) or (c). In the final section, we extend our exhibit of these three origami models to a set of representatives of all 17 wallpaper types, of which 12 are adapted from those of Figure 1.

Throughout, we use the crystallographic, or Hermann-Mauguin, notation for naming the distinct wallpaper groups and associated pattern types. We refer to Fathauer [2, pp:60–65] for descriptions of the groups. The starting point for the three designs of Figure 1 is the tessellation "Five-and-Four," presented as the first beginner project in the book *Origami Tessellations* by Eric Gjerde [3]. This book was my first introduction to this intriguing confluence of art and mathematics, and still serves as a valuable resource.

Tessellations and Their Symmetries

A *periodic tessellation* of the plane is a covering of $\mathbb{R}^2$ by tiles whose interiors do not overlap, and for which there is a bounded patch of tiles that can generate the tessellation by translations in two linearly independent directions. A minimal such bounded patch is a *translation unit cell*. In Figure 1, it is only in (c) that the highlighted square is a translation unit cell for the pattern. In each of our examples, we identify a translation unit cell that is either a square, rectangle, regular hexagon, or rhombus.

An *isometry* of the plane $\mathbb{R}^2$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ that preserves Euclidean distance, and hence faithfully preserves all geometric structure. Isometries include translations, rotations, reflections, and all compositions of such transformations. An isometry f is a *symmetry* of a periodic tessellation $\mathcal{T}$, if applying f to a plane that is decorated with $\mathcal{T}$ fully preserves $\mathcal{T}$. The set of all symmetries of a particular periodic tessellation is a group under composition, known as a *wallpaper group*. Two tessellations belong to the same wallpaper class if their symmetry groups are isomorphic, via an isomorphism that preserves translations, rotations and reflections. We briefly summarize some properties of symmetries of periodic tessellations, and refer to [1] for details and proofs.

Translational Symmetries of Periodic Tessellations

Every vector $\vec{v}$ in the plane defines a *translation* $T_{\vec{v}} : \mathbb{R}^2 \to \mathbb{R}^2$, via $T_{\vec{v}}(x) = x + \vec{v}$, for $x \in \mathbb{R}^2$. The translation $T_{\vec{v}}$ shifts the entire plane $\mathbb{R}^2$ by the vector $\vec{v}$. If $\vec{v}$ and $\vec{w}$ are two vectors, then the composition $T_{\vec{v}} \circ T_{\vec{w}}$ is the translation $T_{\vec{v}+\vec{w}}$. A periodic tessellation is defined by the following property:

> *For every periodic tessellation $\mathcal{T}$ of $\mathbb{R}^2$, there is a pair of linearly independent vectors $\vec{u}$ and $\vec{v}$, for which the translational symmetries of $\mathcal{T}$ are exactly those translations of the form $T_{a\vec{u}+b\vec{v}}$, where a and b are integers.*

The vectors $a\vec{u} + b\vec{v}$, for integers a and b, comprise the *lattice of translations* of $\mathcal{T}$, and its elements move a translation unit cell to cover the entire plane.

Rotational Symmetries: The Crystallographic Restriction

If a periodic tessellation $\mathcal{T}$ possesses a rotational symmetry through an angle θ, then the possible values of θ are limited to integer multiples of 60° or 90°. This limitation is known as the *crystallographic restriction*. Thus, non-identity rotational symmetries of periodic tessellations can only have order $2, 3, 4$ or 6. This remarkable statement prompts the idea of classifying periodic tessellations according to the greatest order of their rotational symmetries. A tessellation possessing a 4-fold rotational symmetry cannot also have a 3-fold or 6-fold rotation with a different centre, since the composition of such operations would violate the crystallographic restriction.

Reflection Symmetries of Periodic Tessellations

The symmetries of a periodic tessellation $\mathcal{T}$ may include reflections.

Glide Reflections of Periodic Tessellations

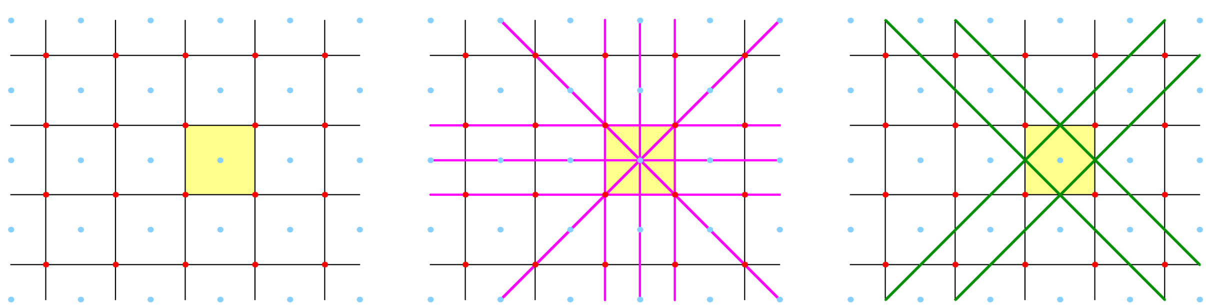
A glide reflection is the composition of a reflection with a translation by a vector parallel to the reflection axis. A glide reflection is of interest, as an element of the symmetry group of a tessellation, when neither the reflection nor translation component is itself a symmetry of the tessellation.

Symmetry elements of a periodic tessellation are rotation centres, reflection axes, and glide-reflection axes of symmetries of the tessellation. Every symmetry of a periodic tessellation must preserve the symmetry elements and their types. This means that every symmetry must transform each rotation centre to a rotation centre of the same order, and must carry reflection axes to reflection axes, and glide reflection axes to glide reflection axes.

Wallpaper Groups with 4-Fold Rotations

We now outline some steps towards the classification of the wallpaper groups possessing rotations of order 4. Suppose that $\mathcal{T}$ is a periodic tessellation of $\mathbb{R}^2$ in which the origin is the centre of a 4-fold rotation. We may assign the coordinate grid so that shortest translations have length 1, and so that $(1, 0)$ is another 4-fold rotation centre, and is the image of $(0, 0)$ under a shortest translation. By applying the 90° rotation about the origin to $(1, 0)$, we find that $(0, 1)$ is another element of the translation lattice, and hence so is every point with integer coordinates. Since no translation has distance less than 1, these points comprise the entire lattice of translations.

Moreover, each point with integer coordinates is a 4-fold rotation centre in $\mathcal{T}$, since it is the image under a translation of the 4-fold rotation centre at $(0, 0)$. The square whose vertices have coordinates $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ is a translation unit cell for the tessellation. This square is highlighted in Figure 2(a) below, where points with integer coordinates are coloured red. The blue points indicate the centres of this distinguished square, and its translates; they have half-integer (but not integer) coordinates. By composing the 90° counterclockwise rotation about $(1, 0)$, after the translation $T_{(1,0)}$, we find that the blue point $(\frac{1}{2}, \frac{1}{2})$ is the centre of a 4-fold rotation in the symmetry group of $\mathcal{T}$, as are all its translates. Any further 4-fold rotation centres would lead to a translation of length less than 1, as a composition of rotations. We conclude that the 4-fold rotation centres of $\mathcal{T}$ form two classes, labelled by the red and blue points. Within each of these classes, all elements are mapped to each other by translational symmetries of $\mathcal{T}$, but no translation of $\mathcal{T}$ moves a blue point to a red one. The features shown in Figure 2(a) are common to all wallpaper patterns with 4-fold rotational symmetries.



(a) *4-fold rotation centres* (b) *p4m reflections* (c) *p4g reflections*

Figure 2: *Possible configurations of reflections for wallpaper groups with 4-fold rotations.*

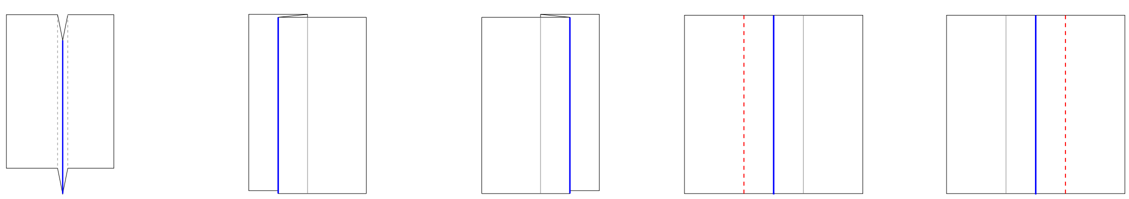
We now consider what reflections might preserve the set of 4-fold rotation centres. We first consider possible reflection axes that preserve the translation lattice (red points). Four such axes intersect at the centre of the highlighted translation unit cell, as shown in Figure 2(b). These reflection axes, along with their translates, define the wallpaper group $p4m$. Patterns of type $p4m$ feature a square translation unit cell with the full dihedral symmetry of the square.

The possibility also exists of a reflection that preserves the system of 4-fold rotation centres in Figure 2(a), but switches the blue points and the red points. This possibility is shown in (c), and determines the wallpaper group $p4g$, which has reflection axes in two perpendicular directions. The 4-fold rotation centres in a $p4g$ pattern do not lie on a reflection axis. The reflection axes in (b) and (c) of Figure 2 cannot occur in the same pattern, since their intersections would force 4-fold rotation centres at the midpoints of the edges of the squares in (a).

A third possibility also occurs, of a symmetry group that admits 4-fold rotations but no reflections. This arrangement describes the wallpaper group $p4$. Although we have not proved that the symmetries of $p4m$, $p4g$ and $p4$ completely classify wallpaper groups with 4-fold rotations, we have identified and distinguished these three possibilities.

Modifying a $p4m$ Origami Model

We now discuss the construction of the $p4m$ model of Figure 1(a), and its adaptation to other wallpaper patterns. To describe the construction and modification processes, we need to define the term *pleat*, in the context of flat folded origami designs in which all folds occur along (segments of) straight lines. In accordance with Lang [4], we define a pleat as a side-by-side mountain and valley fold pair. The construction of a pleat (in the simplest case of parallel folds) is shown in Figure 3. In our crease patterns, a solid blue line represents a mountain fold, a dashed red line represents a valley fold, and a solid grey line represents an unfolded crease. From the configuration in Figure 3(a), we may form a flat pleat by folding to the left or right, which designates one of the dashed grey lines of (a) as a valley fold, and the other as an unfolded crease. We refer to the switch between the left and right configurations as a *pleat reorientation*. As a physical move, it resembles turning a page in a book.



(a) *pleat formation* (b) *left pleat* (c) *right pleat* (d) *left creases* (e) *right creases*

Figure 3: *Pleat formation and orientation.*

In practice, a pleat generally intersects other features of an origami model, and cannot be reoriented independently of its environment. However, pleat reorientations whose wider effects are confined to a local neighbourhood are possible. This idea is the key to our wallpaper pattern modifications, and is an essential design consideration for adaptable wallpaper models. As an example, we now discuss the details of the adjustment of the highlighted square of Figure 1(a) to that of Figure 1(c). We refer to these squares as the $p4m$*-cell* and $p4$*-cell* respectively. Figure 4 shows their crease patterns.

The origami models of Figure 1 (and all others in this article) are hand-folded from a square sheet of paper. For the $p4m$ design, the first step is to precrease a square grid, shown in grey in Figure 4. Each copy of the $p4m$-cell involves an 18×18 region of the precreased grid. The finished model has regions with one, three, five and seven layers of paper, distinguishable in the backlit images of Figure 5 by their different levels of translucency.

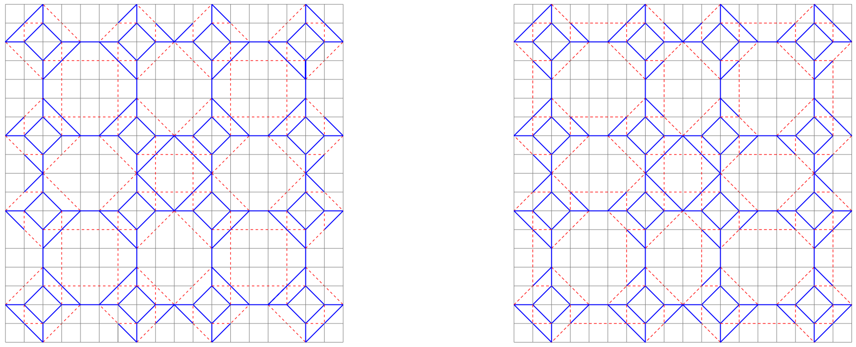
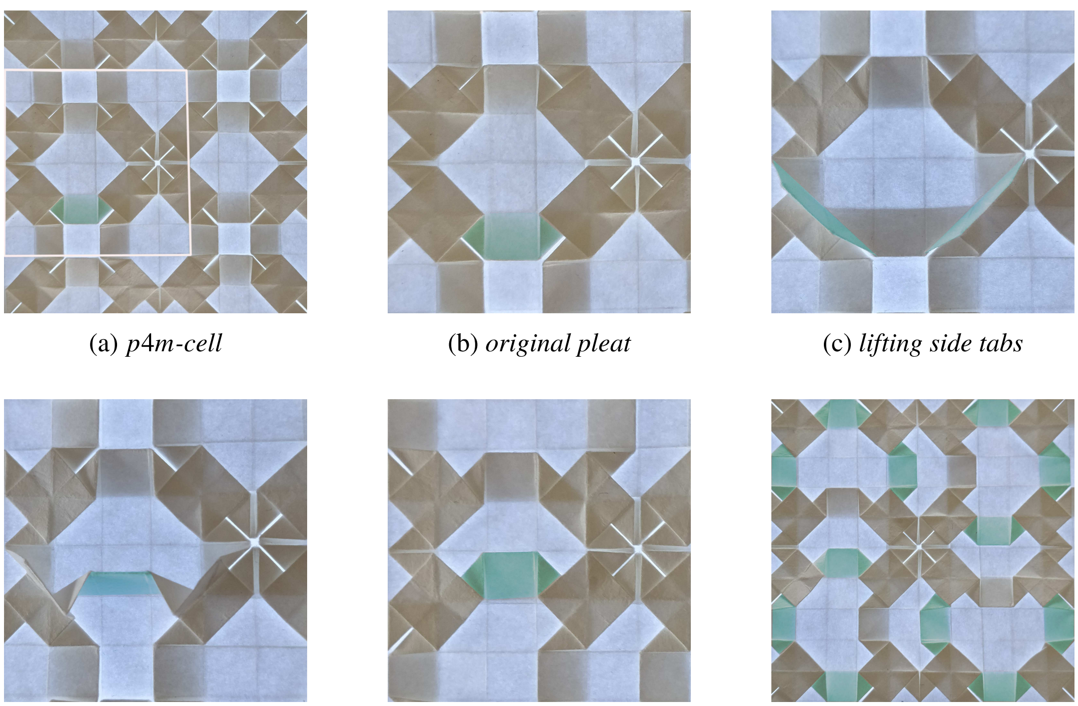


Figure 4: *Crease patterns for $p4m$-cell (left) and $p4$-cell (right).*

The dimly backlit photographs of Figure 5 show some steps in the transformation of the $p4m$-cell to the $p4$-cell, by reorienting twelve pleats of the same kind. Figure 5(a) shows the $p4m$-cell, with an outlined square that is enlarged in (b) through (e). These four images show stages in the reorientation of the pleat that is coloured in (a). Figure 5(f) shows the resulting $p4$-cell, with all twelve reoriented pleats highlighted. Reorientations of the same type, applied to just eight of these twelve pleats in the $p4m$-cell, modify it to the $p4g$-cell that is highlighted in Figure 1(b).



(a) *$p4m$-cell* (b) *original pleat* (c) *lifting side tabs*

(d) *lifting pleat* (e) *repositioned pleat* (f) *$p4$-cell*

Figure 5: *Pleat reorientations in the $p4m$ modifications.*

The reconfigurations presented in Figures 1 and 5 are not the only ones that convert the $p4m$ pattern of Figure 1(a) to a $p4$ or $p4g$ pattern, even if we confine ourselves to pleat reorientations of this kind. For any periodic model that affords some scope for local adjustments, we can consider the possibility of adaptation to a pattern of another wallpaper class. The models in Figure 1(b) and (c) result from considering the following questions about the $p4m$ model of Figure 1(a), in view of Figure 2.

- Whether selected local adjustments can eliminate all reflection axes while preserving 4-fold rotation centres, converting the $p4m$ pattern to a $p4$ pattern.
- Whether selected local adjustments can eliminate reflection axes through the 4-fold rotation centres, and introduce new reflection axes as in Figure 2(c), transforming $p4m$ to $p4g$. In the $p4g$ model of Figure 1(b), the reflection axes are horizontal and vertical lines that outline squares whose centres are the 4-fold rotation centres.

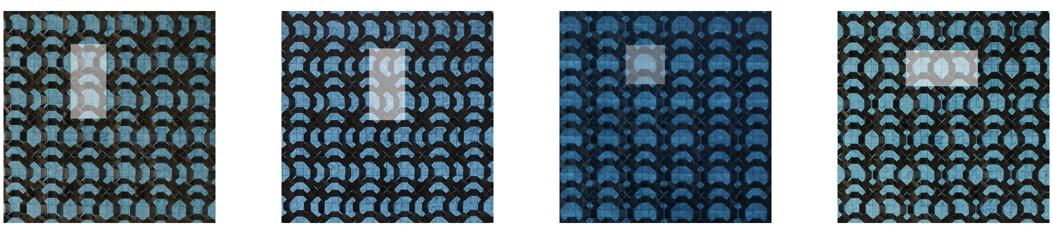
The Rest of the 17 Wallpaper Groups

In this section we present backlit origami models representing the remaining fourteen wallpaper groups, with a translation unit cell highlighted in each. The first nine of these are pictures of the same sheet of blue paper, adapted from the $p4m$ pattern of Figure 1(a), using pleat reorientations of two kinds. The last five, which involve 3-fold rotations, are folded from an isometric triangular grid and adapted using similar ideas, from the model that represents the wallpaper group $p6m$. We organize the 17 patterns (including the three shown in Figure 1) according to the maximum order of their rotational symmetries, and list their additional non-translational symmetries.

No Non-identity Rotations

There are four wallpaper groups whose only rotation is the identity element.

- $p1$ has no additional symmetries.
- pg has no reflections, and has glide reflections with evenly spaced parallel glide reflection axes.
- pm has reflections with evenly spaced parallel axes, and no glide reflections.
- cm has evenly spaced parallel axes that are alternately reflection axes and glide reflection axes.



(a) $p1$ (b) pg (c) pm (d) cm

Figure 6: *Origami models for the four wallpaper groups with maximum rotation order 1.*

For $p1$, pg and pm, a pair of perpendicular sides of the unit translation cell can serve as generating vectors for the lattice of translations. For cm, the vertical side of the highlighted rectangle and a vector directed from the lower left corner to the midpoint of the upper edge comprise a basis for the translation lattice. In the cm pattern, the arrangement of the rectangular translation unit cells resembles the "running bond" pattern of brickwork.

Maximum Rotation Order 2

There are five wallpaper groups whose only non-identity rotations are 180° rotations.

- $p2$ has only 2-fold rotations, no reflections or glide reflections.

- *pgg* has no reflections, and has evenly spaced glide reflection axes in perpendicular directions, with 2-fold rotation centres not on glide reflection axes.
- *pmg* has evenly spaced parallel reflection axes, and evenly spaced glide reflection axes perpendicular to the reflection axes; 2-fold rotation centres lie on glide reflection axes but not on reflection axes.
- *pmm* has reflection axes in perpendicular directions, with all 2-fold rotation centres at intersections of reflection axes.
- *cmm* has reflection axes in perpendicular directions, and separate glide reflection axes parallel to these; 2-fold rotation centres lie at the intersections of reflection axes and at the intersections of glide-reflection axes.

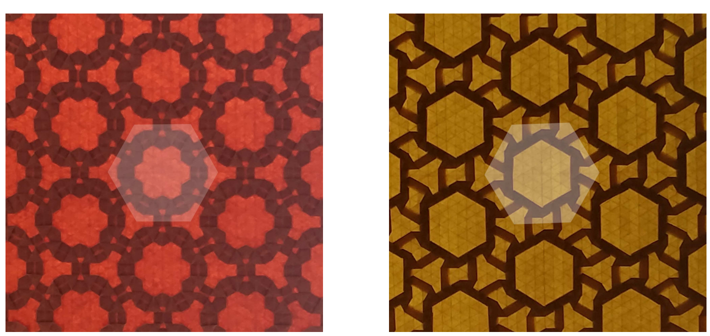
(a) $p2$ (b) pgg (c) pmg (d) pmm (e) cmm

Figure 7: *Origami models for the five wallpaper groups with maximum rotation order 2.*

For each model in Figure 7 except *cmm*, a pair of perpendicular edges of the translation unit cell serve as generating vectors of the translation lattice. For *cmm*, the arrangement of rectangular translation unit cells is similar to that of *cm*, resembling a running bond of brickwork.

Maximum Rotation Order 6

There are two wallpaper patterns that have 60° rotational symmetries. The first of these is $p6m$, and is analogous to $p4m$ for 4-fold rotations. A $p6m$ pattern has a translation unit cell in the shape of a regular hexagon, with the full dihedral symmetry group of a hexagon. Any periodic tessellation that has a reflection axis through a 6-fold rotation centre is a $p6m$ pattern. A tessellation with a hexagonal translation unit cell that possesses 6-fold rotational symmetry, but no reflection symmetry, corresponds to the wallpaper group $p6$. Figure 8 shows closely related origami constructions for $p6m$ and $p6$. The $p6$ pattern is adapted from the $p6m$ pattern by local adjustments that preserve the rotational symmetries, and eliminate the reflections.



(a) $p6m$ (b) $p6$

Figure 8: *Origami models for the two wallpaper groups with maximum rotation order 6.*

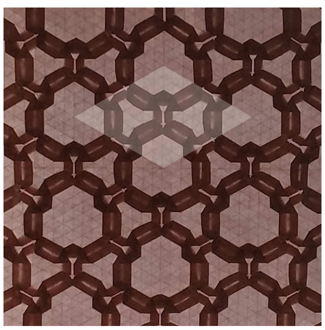
In $p6m$ and $p6$, the lattice of translational symmetries is generated by a pair of vectors, each directed from the midpoint of one side of a hexagonal translation unit cell, to the midpoint of the opposite side.

Maximum Rotation Order 3

There are three further wallpaper classes whose symmetries include 120° rotations. Each can be realized by adjustments to the origami model for $p6m$, that eliminate certain reflection axes, converting the 6-fold rotation centres of the $p6m$ model to 3-fold rotation centres.

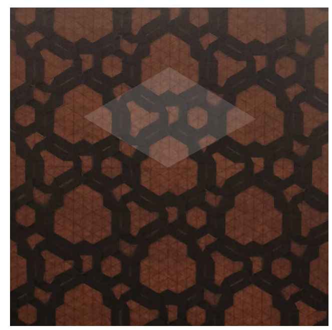
- $p31m$ has a translation unit cell consisting of two equilateral triangles that share an edge; reflection axes for the tessellation lie along the three edges of both triangles. The 3-fold rotation centres occur at the vertices of the triangles (where reflection axes intersect), and at the centres of the triangles, which do not lie on reflection axes.
- $p3m1$ has a translation unit cell consisting of two equilateral triangles that share an edge; each of the triangles has a pattern admitting full dihedral symmetry, but the triangles' patterns are not mirror images of each other. In a $p3m1$ pattern, all 3-fold rotation centres lie on reflection axes.
- $p3$ has a translation unit cell consisting of two equilateral triangles that share an edge, where the pattern within each triangle has 3-fold rotational symmetry only. A $p3$ pattern has no reflection symmetries.

In each of $p31m$, $p3m1$ and $p3$, a pair of adjacent edges of the rhombus-shaped translation unit cell determines vectors that generate the lattice of translations.



(a) $p31m$

(b) $p3m1$



(c) $p3$

Figure 9: *Origami models for the three wallpaper groups with maximum rotation order 3.*

Summary and Conclusions

The article aims to demonstrate that the art form of origami tessellations provides a structure to facilitate an absorbing practical exploration of the relationships between the symmetry systems characterized by the wallpaper groups. By making a local adjustment to a repeating element, one can observe the impact on the pattern's symmetry group. Origami is an ideal environment for exploring this phenomenon, because it allows the possibility of implementing the local adjustments as reversible physical moves.

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