Constructing Wooden Polyhedra with a Sander

George Hart

Supplement to the paper “Constructing Wooden Polyhedra” at the Bridges 2022 conference

For safety reasons, power saws may be problematic in school environments, so I have considered other ideas for student activities. A method I have not seen discussed before is based entirely on a powered disc sander. Although careful shop safety practices must still be followed, a sander is much safer than a saw, so may be suitable for novice woodworkers. A table saw may be used by the instructor to prepare plain cubes before-hand; they are the raw material for students to use with the sander. Scraps from construction 2×4s are easily pre-cut into sandable 1.5 inch cubes. This is a suitable size for making most of the Archimedean solids, but for some of the most intricate models, this is too small. For the most complex ones, e.g., the snub dodecahedron and great rhombicosidodecahedron, a larger starting cube is recommended.

I have not been able to test this activity yet with students, but from experiments in my home shop I expect it can be made to work. I use a standard hobbyist-quality bench-top sander which combines a 4×36-inch belt with a 6-inch disc, affordably available at many hardware stores and home outlets. Many variations on this should work equally well. I find a disc sander to be more accurate than a belt sander, but you may find these methods also work well with a belt sander.

A typical sander has a disc in a vertical plane behind a table with an adjustable tilt. A miter gauge sits in a slot in the table to allow a precise rotation in the plane of the table. The combination of table tilt and miter angle rotation is like longitude and latitude, allowing the user to orient any desired point of the work-piece against the sanding disc. Knowing the angles in space relative to the faces that rest against the table and miter gauge, one can sand away material to create a new face in any chosen orientation.

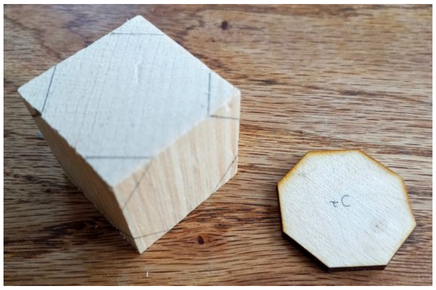
To determine the depth of material to remove, pencil marks can first be made on the work-piece by tracing a template. Figure A shows a marked 1.5 inch cube and the octagonal template used to make the simplest example, a truncated cube. Figure B shows how it sits properly oriented on the disc sander. The table is tilted down 35.26 degrees and the block is rotated 45 degrees relative to the table using the miter gauge. (Or one could use a guide piece that is clamped to the table and serves the same purpose, as illustrated below.)

The key angles are straightforward to work out for a cube, as shown in Figure C. The notation is chosen so $\mathbf{P}_i$ is the point at the center of an *i*-dimensional component, i.e, $\mathbf{P}_0$ is a vertex, $\mathbf{P}_1$ is an edge midpoint, $\mathbf{P}_2$ is a face center and $\mathbf{P}_3$ is the body center. (Together they determine a small tetrahedron called an “orthoscheme” because each of its four faces is a right triangle. A cube can be dissected into 48 congruent orthoschemes that all touch $\mathbf{P}_3$.) For a cube of edge 2, the six edge lengths are directly worked out via the Pythagorean theorem to be $\mathbf{P}_0\mathbf{P}_1$=$\mathbf{P}_1\mathbf{P}_2$=$\mathbf{P}_2\mathbf{P}_3$=1, $\mathbf{P}_0\mathbf{P}_2$=$\mathbf{P}_1\mathbf{P}_3$=Sqrt[2], and $\mathbf{P}_0\mathbf{P}_3$=Sqrt[3]. Then basic trigonometry gives the three essential angles we need between the surface points, as seen from the center of the cube: $\mathbf{\Theta}_{132}$=ArcSin[1/Sqrt[2]]=45, $\mathbf{\Theta}_{031}$=ArcSin[1/Sqrt[3]]=35.26, and $\mathbf{\Theta}_{032}$=ArcSin[Sqrt[2]/Sqrt[3]]=54.74 degrees. (Examine a cube standing on edge to see why $\mathbf{\Theta}_{031} + \mathbf{\Theta}_{032} = 90$.)

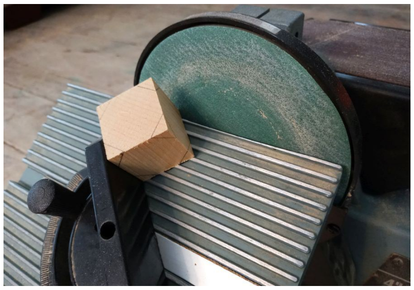
With a trickier derivation [4, 11], the analogous central angles for a regular dodecahedron are worked out to be $\mathbf{\Theta}_{132}$=31.7, $\mathbf{\Theta}_{031}$ =20.9, and $\mathbf{\Theta}_{032}$ =37.4 degrees. (For the dodecahedron or icosahedron, there are 120 congruent orthoschemes, all meeting at $\mathbf{P}_3$. Examine a model standing on edge to see the beautiful fact that $\mathbf{\Theta}_{132} + \mathbf{\Theta}_{031} + \mathbf{\Theta}_{032} = 90$ on a dodecahedron or icosahedron.) I recommend Zometool for developing strong visualization skills and intuition about the geometry of dodecahedra and other polyhedra with icosahedral symmetry [11]. For the Archimedeans with icosahedral symmetry, one can first make a dodecahedron or icosahedron by sanding to the planes of Sharp's method, then use templates and these tilt and miter angles for truncating corners and chamfering edges as needed. Figure D shows a dodecahedron marked with a decagons template for making the great rhombicosidodecahedron of Figure 9b in the paper. A guide piece cut for the icosahedron or dodecahedron dihedral angle allows one face to sit flat on the table and another face to sit against the guide piece. Figure E shows the guide piece for positioning a dodecahedron (marked with the decagon template shown), about to be sanded to make a truncated dodecahedron. The guide piece (cut for the 116.5 degree dihedral) is clamped to the table an angle of 18 degrees and the table is tilted down 21 degrees.

To mark the faces for the proper depth of sanding, templates can be precut with a laser cutter. The construction dimensions for the templates may be found in many references [12, 18]. From these fundamental ideas, all the Archimedean solids can be worked out. Almost all of them have all their face planes normal to the three types of radial segments in the orthoscheme, i.e., $\mathbf{P}_3\mathbf{P}_0$, $\mathbf{P}_3\mathbf{P}_1$, and $\mathbf{P}_3\mathbf{P}_2$, so the tilt and miter angles to use are easily worked out from the three central angles given above.

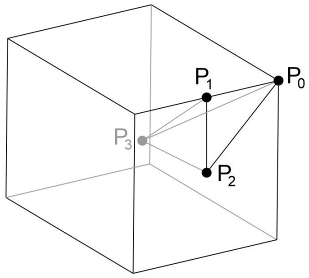
The chiral Archimedeans (the snub cube and snub dodecahedron) are exceptions because they have a family of faces in a unique set of skew planes. For them, the easiest process is to start with a cube or dodecahedron, respectively, trace the template on all the faces, sand the faces normal to $\mathbf{P}_3\mathbf{P}_0$, then finally sand the skew planes. (The same sequence of operations is described by Nakagawa, but with a saw and special positioning jig [17].) To clearly see the depth for sanding the skew planes, mark the corner faces with pencil lines. The snub dodecahedron template (Figure F) has an inner pentagon scaled to 56.2%, rotated 13.1 degrees. The snub cube template has an inner square scaled to 43.8%, rotated 16.5 degrees [18]. Flipping the template over switches between the left-hand and right-hand enantiomorph. Be sure to keep just one side up for any one model.



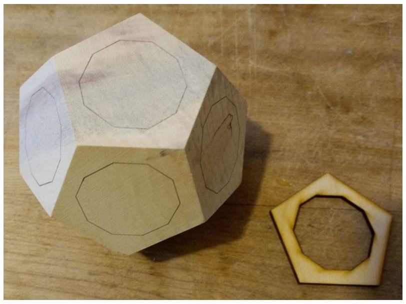
A. *Template for truncated cube.*



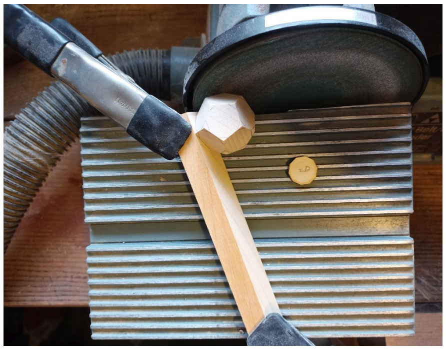
B. *Marked block on disc sander.*



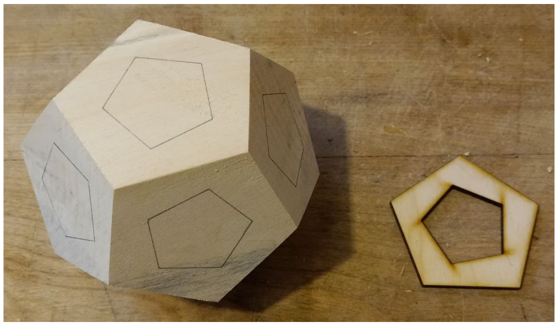
C. *Essential polyhedral points.*



D. *Template for great rhombicosidodecahedron.*



E. *Guide piece for truncating dodecahedron.*



F. *Template for snub dodecahedron.*