Modular Origami Map Coloring Models

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Abstract

In this workshop we will learn how to construct a model of a map on a torus with seven regions using modular origami with Sonobe units. Each region of the map shares a border with the other six, so seven colors are required to properly color the map. This paper also includes instructions for a second model based on the Ungar-Leech seven-color torus and a model of a map on a double torus requiring eight colors.

Inspiration

The Four-color Theorem states that any map on a plane or a sphere can be colored with four colors so that regions that share a boundary are different colors. This was first conjectured in 1852 by Francis Guthrie, yet the theorem wasn't confirmed until Kenneth Appel and Wolfgang Haken provided a computer aided proof in 1976. While it is easy to draw a map on a sphere that requires four colors, maps showing the maximum number of colors needed on other surfaces are more complex. In 1890 Percy Heawood published a map on a torus requiring seven colors and showed seven colors are sufficient for any map on this surface. He also showed that eight colors were sufficient for maps on a two-holed torus but neglected to show such a map exists. The following year, Lothar Heffter showed there is a map on the double torus that requires eight colors.

Many mathematical artists have created models of maps on a torus that require seven colors. Moira Chas' online AMS column [3] shows three distinct maps on crocheted models. Susan Goldstine and Ellie Baker include patterns for several bead crochet bracelet models in their book [1]. They also published directions for a model made of large plastic pony beads [7]. Ellie Baker and Kevin Lee designed several toroidal silk scarves where the seven regions are identical images of a bird that form a figurative tessellation of the surface [2]. Goldstine curates a website [6] that includes her seven-color ceramic torus coffee mug, her model sewn out of fleece fabric, and her poster-board model based on the seven-sided Szilassi polyhedron. This site also includes images of a hydrostone model by Norton Starr, a crocheted model by

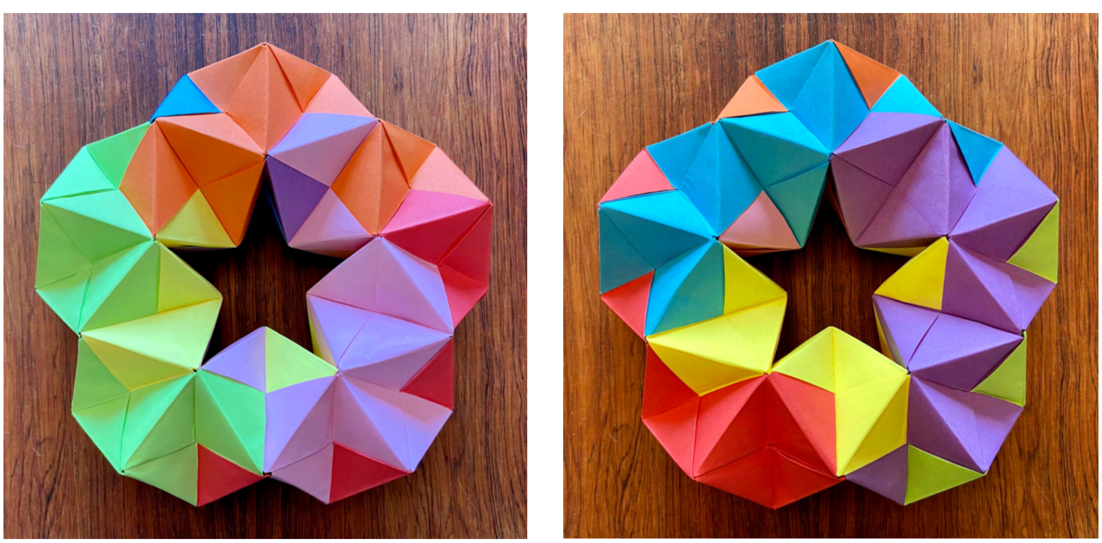


Figure 1: *A model of a seven-color torus constructed from 45 Sonobe modular origami units.*

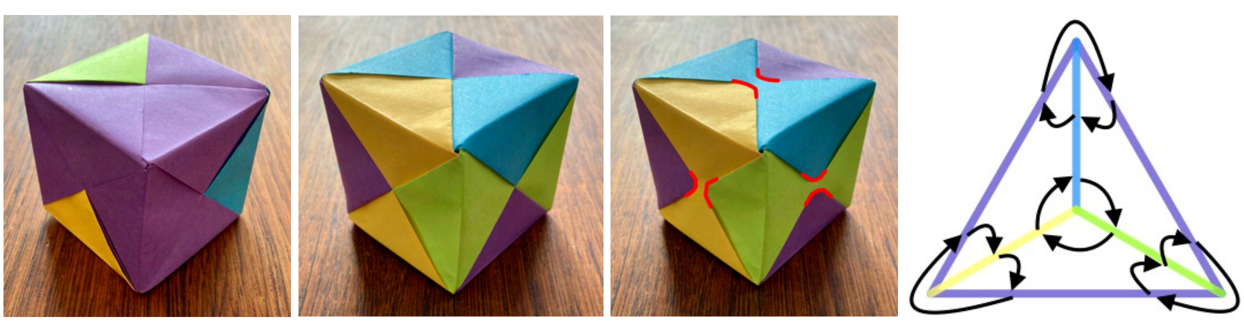
Carolyn Yackel, a fabric color changing belt model by Skonna Britain, a silk scarf by Ellie Baker, and a model made by Faye Goldman from over 3200 strips of ribbon using the Snapology technique developed by Heinz Strobl. Most of these models took considerable time and expertise to design and construct. The model in Figure 1 is made from modular origami using Sonobe units and can be constructed without special materials or skills. In this workshop we will build this model and learn about the two other models in this paper.

Models from Sonobe Units

In modular origami many identical units are combined to form a geometric or abstract figure. Sonobe units [10] are one of the simplest and most flexible modular units and can be used to build many shapes. But Sonobe units are a crude medium and thus require some compromises when trying to make mathematical models. The models in this paper require us to think abstractly about what both the surface and the map coloring represent. These models are not exact representations but are topologically equivalent to the objects they model. We will discuss these issues by examining a modular origami cube and explaining how it can serve as a model of a four-colored map on a sphere.

A cube and a sphere are topologically equivalent, meaning the cube can be continuously deformed, without cutting or puncturing, into a sphere. The cube in Figure 2 is constructed from six Sonobe units as in [11]. There are 3 purple units and one each in yellow, blue, and green. The three purple units form a single region on the back of the cube. Notice that the yellow, blue, and green regions visible on the front of the cube each consist of two squares that meet at a point. Each square is bent in half around an edge of the cube. In the theory of map coloring a single region is not allowed to have parts that meet only at a point [13]. Instead, regions that meet at a point can represent different countries that can be colored the same color. In our Sonobe models we will allow a single region to contain parts that meet at a point as long as we can theoretically enlarge that connection without compromising the coloring of the map. In Figure 2c the yellow, blue, and green regions have been digitally deformed between the red segments to show how they represent proper map regions. Notice that each of the four colored regions still touches the other three.

We can use a graph to visualize the structure of these models. The graph in Figure 2d represents the four-color cube (i.e. sphere). Each Sonobe unit is represented by an edge in the graph. The graphs for each model have been designed with the assumption that units are always joined by inserting a unit into the unit to the right in a clockwise direction. The arrows around each vertex indicate this direction of insertion. While it is possible to connect Sonobe units in a counter-clockwise direction, these models have been designed assuming the clockwise insertion. Inserting any of the modules in a counter-clockwise direction may cause the coloring to fail. It is very important to keep this convention in mind while building the models; it is easy to forget.

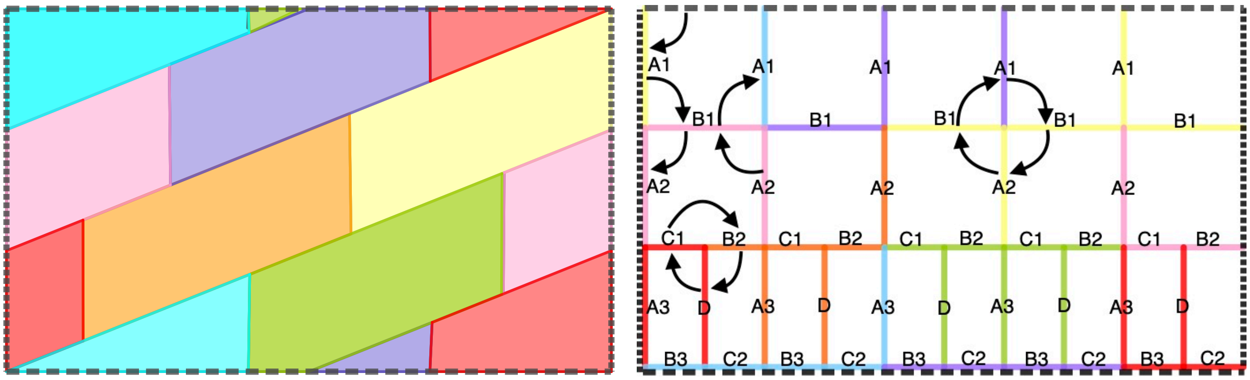


(a) *(b)* *(c)* *(d)*

Figure 2: *(a) A six Sonobe unit modular origami cube in four colors, back view. (b) the same cube viewed from the front. (c) An image of photo (b) digitally altered to show how regions can be thought of as satisfying the traditional rules for maps. (d) A graph showing how to assemble 6 Sonobe units to make a four-color model of a "sphere"*

Building a Seven-Color Sonobe Torus

The design of the first Sonobe torus model was derived from the torus map in Figure 3a. Identifying the dashed lines at the top and bottom of the rectangle will form a cylinder. Then identifying the left and right sides of the cylinder (the dotted sides of the rectangle) forms a torus. The graph in Figure 3b shows how to assemble the units in a modular origami model that approximates the map in Figure 3a. Each edge in the graph in Figure 3b represents a Sonobe unit in the model.



(a) (b)

Figure 3: *(a) A map on a torus requiring seven colors. (b) A graph showing how to assemble Sonobe units to make a model of this map.*

Notice how the top half of the graph in Figure 3b and the assembly of units in Figure 4 correspond. Each unit is inserted into the unit to the right clockwise in the graph as indicated by the sample arrows. Before building this model, it is extremely helpful to watch the 13-minute video by Jason Kemppainen [8] showing how to construct a similar torus. The coloring is different for the model in the video but the structure and building process is the same. These directions are also in Kemppainen's book [9].

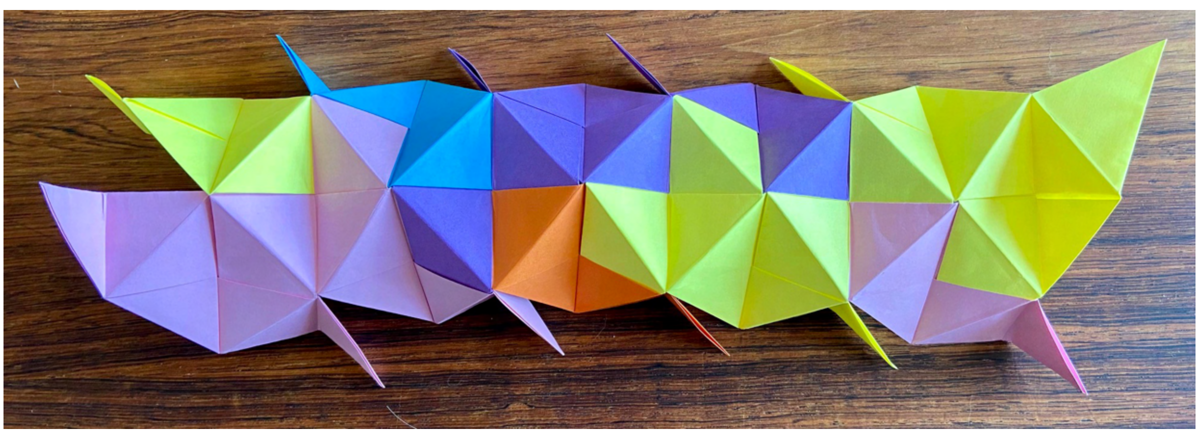
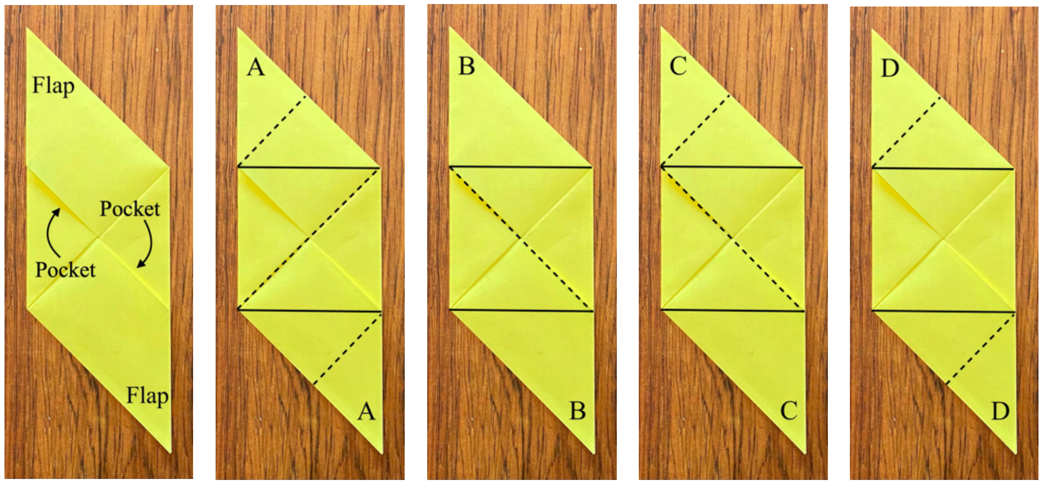


Figure 4: *The assembly of all the Sonobe units labeled A1, B1, or A2 in Figure 3b.*

This seven-color torus is built out of 45 Sonobe units: six units in pink, orange, blue, and yellow and seven units in red, purple, and green. Diagrams for folding Sonobe units can be found online [10] and instructional videos [12] and many books contain directions. A basic Sonobe unit is shown in Figure 5a. Four simple variations of this unit are needed for this model and they are shown in Figures 5b – 5e. Mountain folds are indicated by solid lines and valley folds are dashed lines in the figures. Each edge of the graph in Figure 3b is labeled with the type of unit, A – D, and most edge labels also include a number that helps to further distinguish the Sonobe unit's location in the structure. Labeling each unit with its type (such as A1) at the end of both flaps is very helpful when following the graph to assemble the torus.



(a) (b) (c) (d) (e)

Figure 5: *(a) A basic Sonobe unit. (b) Folds for A units. (c) Folds for B units. (d) Folds for C units. (e) Folds for D units. Mountain folds are solid lines and valley folds are dashed lines.*

Begin construction by inserting the yellow A1 unit into the pink B1 unit as shown on the left of the top row of the graph. Next insert one end of the pink B1 unit into the first pink A2 unit and the other end into the blue A1 unit. Continue to follow the graph until all the A1, B1 and A2 units have been joined as in Figure 4.

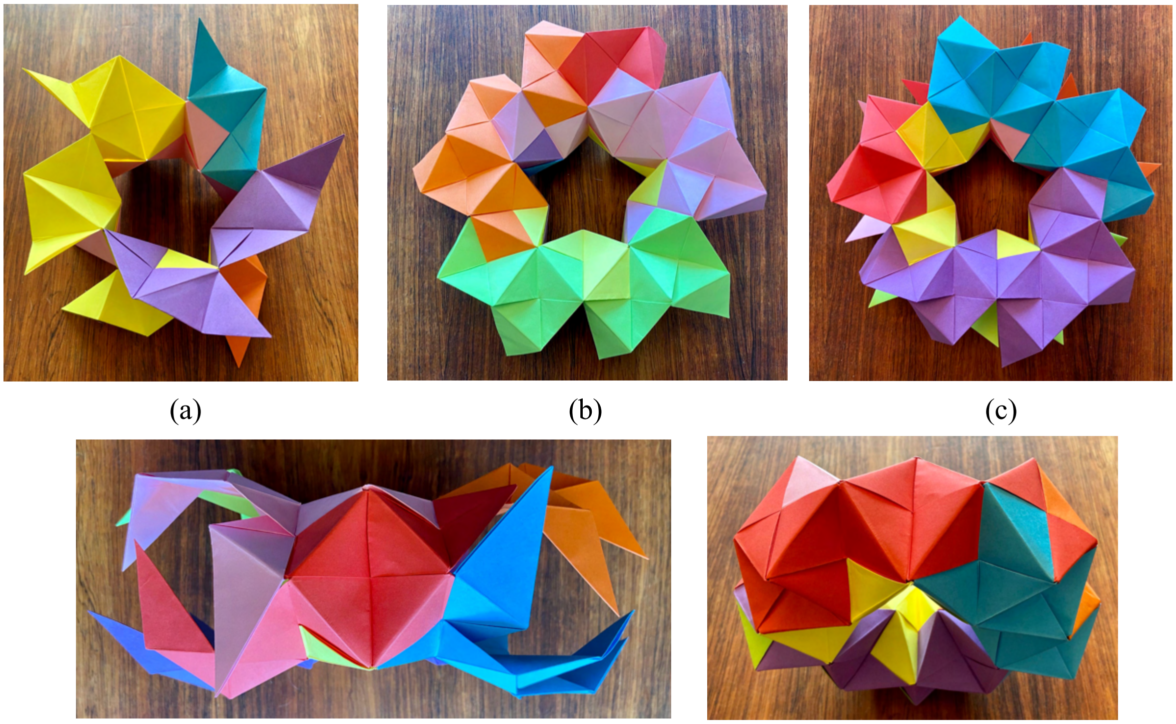
Remember the rectangular graph in Figure 3b represents a torus. Units on the right of the rectangle will be inserted into units on the left side of the rectangle and units at the bottom of the rectangle will be inserted into units at the top (and vice versa). This structure becomes clear as we bend the assembly in Figure 4 into a cylinder that will form the inner ring of the torus as shown in Figure 6a.

While these models can be made without adhesives, it may be helpful to tape together the pieces in Figure 4 on the back before forming the inner ring. Pinching each of the creases to be sure it will fold correctly when joined is also helpful. Bend the assembly so that the rightmost corner of the last yellow B1 unit touches the leftmost corner of the pink B1 unit. Then insert the first pink A2 unit into the last yellow B1 unit and the last yellow B1 unit into the first yellow A1 unit. Notice that this continues to follow the convention of inserting units in the graph in the clockwise direction.

The next step is to add the B and C units. First insert each of the C1 units into the B2 unit to its immediate right in the graph. Taping together these pairs of units on the back may be desirable. Orient the inner ring of the torus so the A2 units are all on top. We are now following the graph from the bottom, so you may want to turn over the graph. Insert the middle pink A2 unit into the red C1 unit and then insert the orange B2 unit into the pink A2 unit on the left. Continue to insert the C1/B2 pairs into the A2 units following the graph. The resulting structure is shown in Figure 6b.

Next join the paired C2 and B3 units in the last row of the graph in the clockwise direction by inserting C2 units into B3 units. Turn over the torus assembly (and the graph) so that the A1 units are on top. Then insert the blue A1 unit into the first blue C2 unit and the first blue B3 unit into the yellow A1 unit to the left. We are inserting the units in the same clockwise direction while identifying the top and bottom edges of the graph. The structure should resemble Figure 6c when all these units have been added.

All that is needed is to complete the outer ring of the torus by adding the D and A3 units following the graph. It is easier to insert the folded flaps for each of these units before straight flaps. Be sure to pull out flaps that get stuck under these units as the model is assembled. Figure 6d shows the torus after the first red A3 piece has been inserted and Figure 6e shows the completed torus from the side. The finished torus should resemble Figure 1. Check that every color touches the other six!



(a) (b) (c) (d) (e)

Figure 6: *(a) The inner ring of the torus as formed from the assembly in Figure 4. (b) The structure after the C1 and B2 units are added. (c) The structure after the C2 and B3 units are added. (d) One red A3 unit added. (e) View of some D and A3 units on the finished torus.*

The Ungar–Leech Seven-Color Torus

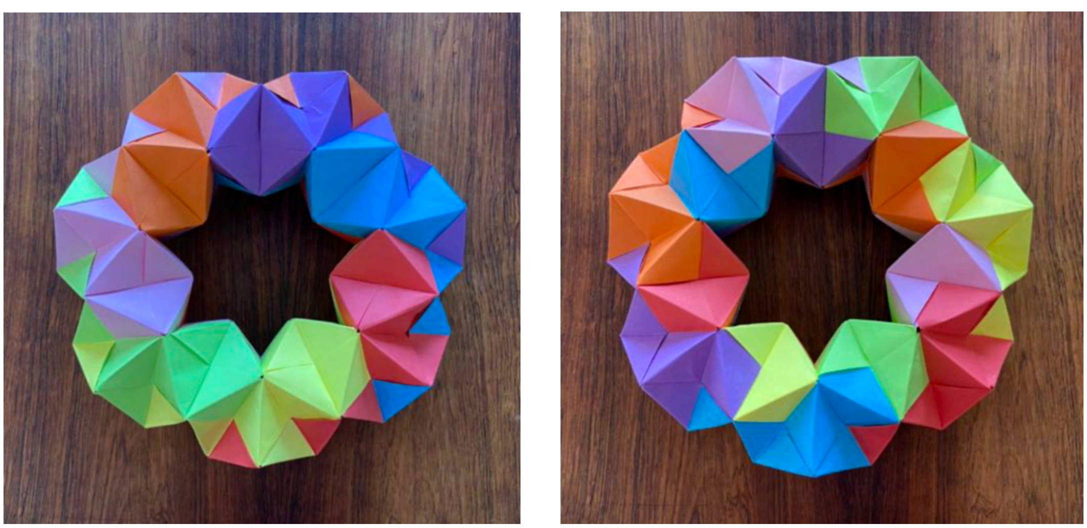


Figure 7: *A model of the Ungar-Leech seven color torus constructed from 63 Sonobe units.*

The Ungar-Leech seven-color torus map was first described by John Leech in 1953 [4]. This Sonobe model is more symmetric than the first model. Every region is an identical assembly of nine units and it can be checked that every color touches every other color by examining just one side, the right image in Figure 7. But this model is more challenging to construct out of Sonobe units.

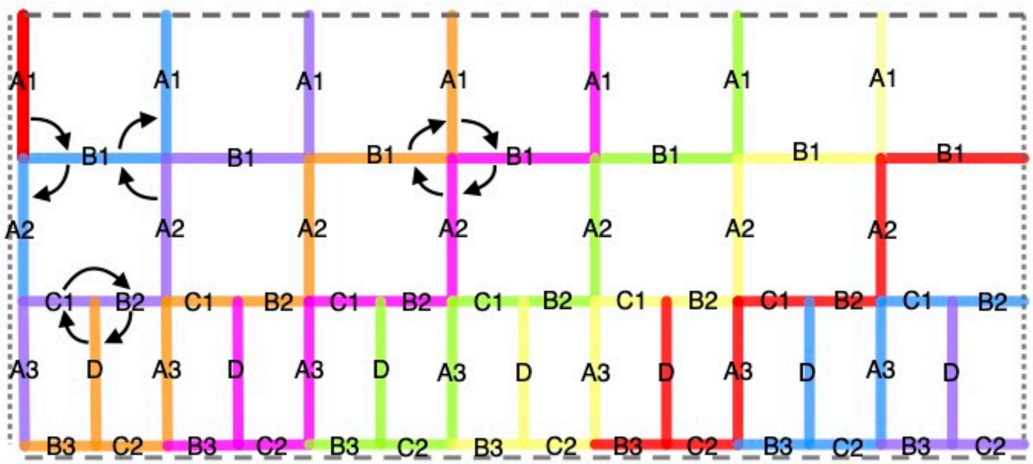


Figure 8: *Graph showing the structure and labeling for the Ungar-Leech model.*

Figure 9: *The assembly of all the A1, B1, and A2 Sonobe units for the Ungar-Leech model.*

The graph for this model is shown in Figure 8 and the assembly of the first three rows is shown in Figure 9. This model is built out of 63 Sonobe units – nine units in each of the seven colors. For each color there are three A units, three B units, two C units, and one D unit. Construction of this model follows the same process as the first model. But because this model uses a larger ring (notice the seven-point star in the center of this model versus the five-point star of the first model) it stresses the natural geometry of the Sonobe units. Depending on the thickness of the paper, the model may get very tight as the last pieces are added. Reshaping the torus will allow room and once the last pieces are added the structure will settle into a nice round shape.

Eight-Color Double Torus Models

Moira Chas has a very nice crocheted eight-colored double torus at the end of her online AMS column [3]. This is a great image to consult to understand the structure of the Sonobe eight-color double torus in this paper. Susan Goldstine's 2014 Bridges paper [5] discusses map coloring on a double torus and the design of her beautiful bead crochet pendant, "Eight-Color 8." It also has images of her eight-color tea set. Designing an eight-color double torus is considerably more complex than a seven-color torus.

The eight-color double torus model in Figure 10 is made from 100 Sonobe units – 20 units in pink and dark green, 17 in orange and purple, 7 in light green and red, and 6 in blue and yellow. The two tori were assembled separately and then joined. The graphs for the two halves of the eight-color double torus are shown in Figure 11. The edges are labeled similarly to the seven-color tori but there are five additional variations of Sonobe units shown in Figure 12 that are used where the two tori are joined.

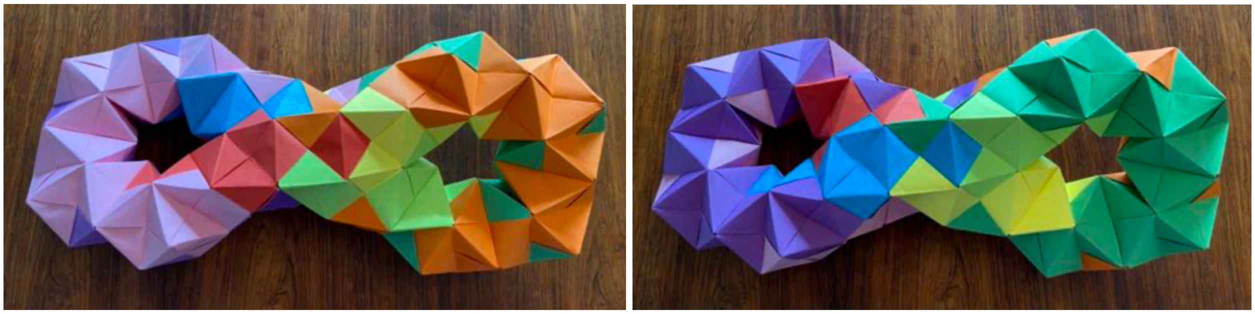


Figure 10: *Two sides of an eight-color double torus made from 100 Sonobe units.*

The graphs in Figure 11 are drawn slightly differently from the graphs in Figure 3b and 8 to accommodate the units that join the two tori; the A1 row in each graph has been shifted to the middle instead of being on the top of the graph. Once both halves have been constructed, they are connected at the units in the middle at the top and bottom of the two graphs. Vertices that are the same color on the graphs are the same vertex in the completed model and show how the two tori line up where they are joined.

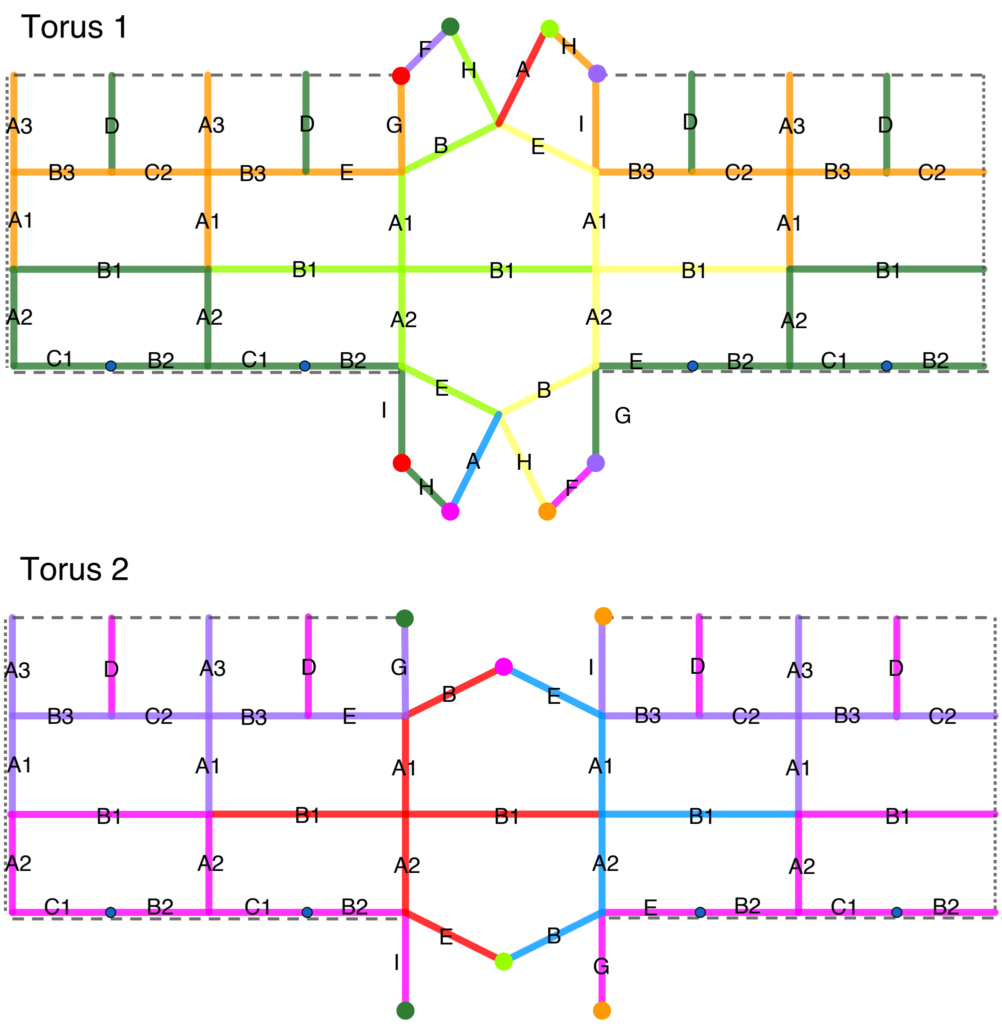


Figure 11: *Graphs showing the structure and unit labeling for the eight-color double torus. Vertices that are colored the same in the graphs will be the same vertex when torus 1 and torus 2 are combined to make the double torus in Figure 10.*

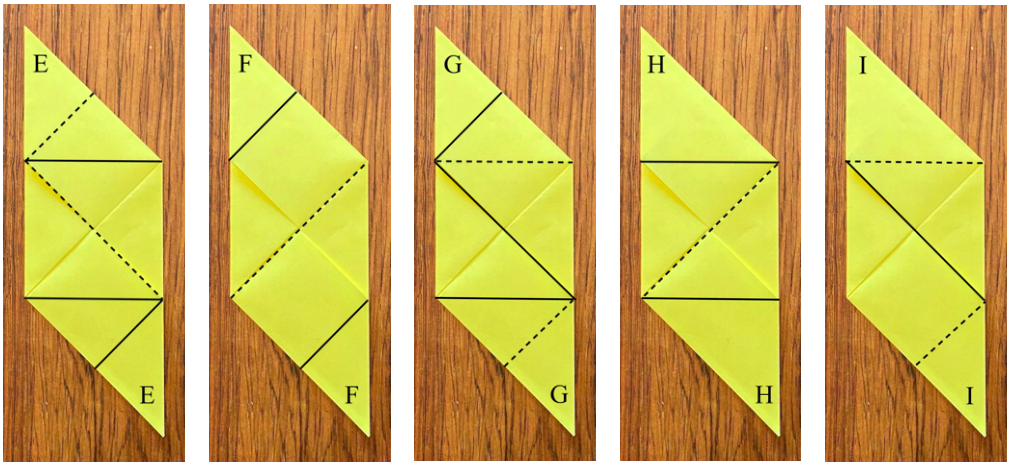


Figure 12: *Additional Sonobe unit variations used for joining the eight-color double torus. Mountain folds are solid lines and valley folds are dashed lines.*

Conclusion

There is no substitute for examining a map model on a surface while turning it in your hands to check how each region touches all the others. And nothing makes the structure clearer than building your own model!

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