Growth Forms of Grid Tilings

Peter Hilgers[1] and Anton Shutov[2]

[1]Hasenbergweg 9, 76275 Ettlingen, Germany; peter.hilgers@posteo.de
[2]Vladimir State University, Gorky Str. 87, Vladimir, 600000, Russian Federation; a1981@mail.ru

Abstract

Growth forms of tilings are an interesting invariant of tilings. They are fully understood in the periodic case but we know very little in the quasiperiodic case. Here we study this problem for quasiperiodic tilings obtained by the grid method. We show that such tilings have polygonal/polyhedral growth forms that result from projections of central sections of orthoplexes. Finally we study characteristics of the computed growth forms in 2D and 3D cases.

Introduction

A **tiling** *T* is a covering of plane or space by finitely many types of polygonal or polyhedral figures which overlap only on their boundaries. There are various methods to construct non-periodoc tilings, here we use the **grid method** which was introduced by de Bruijn [3] in the case of the Penrose tiling. It was widely generalised to obtain 3D quasiperiodic tilings with icosahedral and arbitrary symmetry. The main advantage of the method is that it allows to construct not only point sets, but to find tiles directly. This is beneficial especially in the 3D case.

Grid tilings

We will use the following construction: Let $\{g_1,\ldots, g_N\}$ be a family of N unit vectors in d-dimensional Euclidean space. Choose also N real parameters γ_i which serve as phase shifts. The N-grid L_N is a union of N arrays of equidistant parallel hyperplanes in R^d:

$$L_N = \{x \in R^d : (x, g_i) - \gamma_i \in Z\} \qquad \text{(step 1)}$$

Here (•,•) is a scalar product and $1 \le i \le N$. If there is no point where more than d grid hyperplanes intersect, the grid is called **regular**. Hereafter we always suppose that the grid is regular. Note that the grid will be regular for almost all values of γ_i.

The N-grid L_N defines some tiling of the d-dimensional space, but this tiling is "bad" because the number of tile types is infinite. A key idea of the grid method is to consider the tiling dual to L_N in some sense. Define N functions K_i and function K as follows:

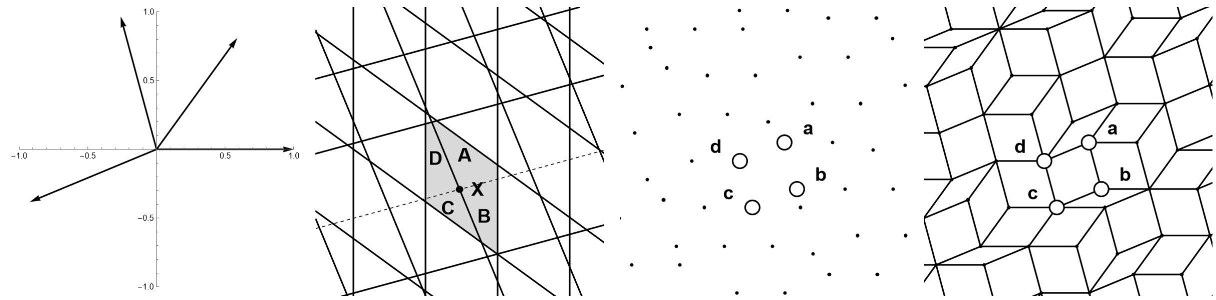
$$K_i(x) = \min\{n \in Z : n \ge (x, g_i) - \gamma_i\} \qquad \text{(step 2)}$$

$$K(x) = \sum_{i=1}^{N} K_i(x) g_i \qquad \text{(step 3)}$$

Informally $K_i(x)$ is the index of the hyperplane perpendicular to g_i through x. It can be proved that $K(x)$ is constant on tiles of L_N. So, K maps the set of tiles of L_N to a discrete set Λ in R^d. The set Λ is a set of vertices of some d-dimensional tiling *Til* that is called **grid tiling**. To define this tiling we must describe edges connecting points from Λ. The rule is as follows: Two points of Λ are connected by an edge if and only if corresponding tiles of L_N have a common edge (step 4).

Furthermore, the set of all tiles of L_N sharing some fixed vertex is mapped (under K) to the set of all vertices of some tile of the grid tiling. In the 3D case the grid tiles are parallelepipeds.

The construction of a 2D tiling is exemplarily illustrated in **Figure 1** for N = 4 and d = 2:



(a) *(b)* *(c)* *(d)*

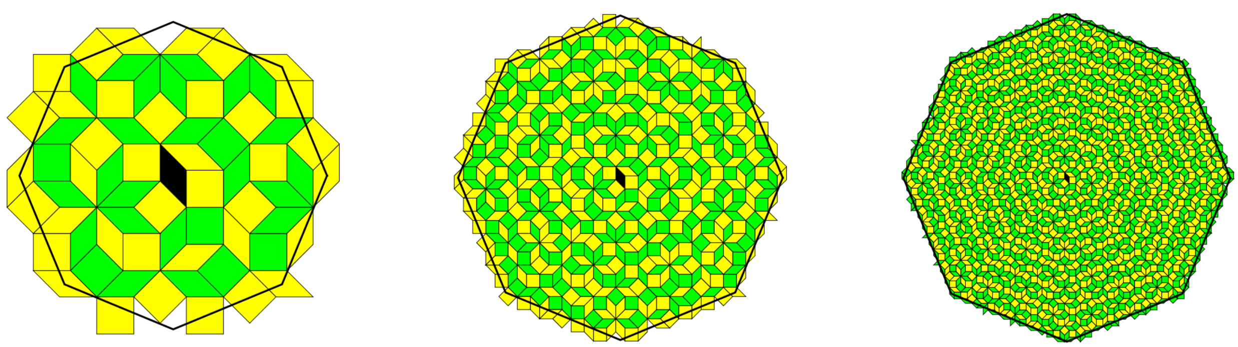
Figure 1: *Construction of a grid tiling: (a) grid vectors g_i, (b) 4-grid L_N (step 1) with intersection point X and adjacent tiles A, B, C, D; dotted line for hyperplane through X and perpendicular to g_3, (c) vertices Λ of the dual tiling (steps 2, 3) with A⟷a, B⟷b, C⟷c, D⟷d, (d) edges of dual tiling (step 4)*

Coronas and Growth forms

A patch P_0 is a finite set of tiles in *Til*. The first **coordination shell** P_1 of P_0 consists of all tiles which are adjacent to a tile of P_0. In the n-th coordination shell P_n are all tiles adjacent to P_{n-1} that are not in P_{n-2}. Here we abstain from giving a formal definition of **growth form**, also known as **corona limit**, and instead refer to [4]. Informally the definition may be given as "outer contour, scaled by 1/n", or

$$\lim_{n\to\infty} \frac{P_n}{n}$$

if it exists, compare **Figure 2**.



(a) (b) (c)

Figure 2: *Initial patch P_0, growth form of Ammann-Beenker tiling (black line) and (a) n=5 corona, (b) n=15 corona, (c) n=30 corona; all tiles scaled by 1/n.*

It is known that all periodic tilings do have a growth form, which is some polygon or polyhedron [2], and we know that the growth form does not depend on the selected initial patch. For non-periodic tilings so far we have only few concrete examples of the calculation of the growth form, such as for the Penrose or Ammann-Beenker tiling [1, 7, 8]. We have not found examples of 3D growth forms in the literature. Many crystallographic applications of growth forms are mentioned in [4]. A complex 3D growth form is the topic of the artwork "Late arrival" [6].

New Theorem and Method to Calculate 2D and 3D Growth Forms

Now consider a regular N-grid tiling *Til* produced by vectors $\{g_1,..,g_N\}$. Let $\{e_1,..,e_N\}$ be the standard orthonormal base of N-dimensional space. Let O_N be a boundary of the convex hull of the vectors $\pm e_i$. The polytope O_N is known as N-dimensional **cross-polytope** or as N-dimensional **orthoplex**. Consider the projections

$\pi_1 : R^N \to R^d$ defined as $\pi_1(t) = \pi_1((t_1,...,t_N)) = \sum_{i=1}^{N} t_i g_i$ and

$\pi_2 : R^N \to R^{N-d}$, an orthogonal projection to the (N-d)-dimensional plane $\pi_1(t) = 0$.

Let P be the d-dimensional plane $\pi_2(t) = 0$. Then for any regular grid tiling the growth form exists and is given by $\pi_1 (O_N \cap P)$. For proof and computational details see [4].

Characteristics of the Growth Forms

The growth forms are centrally symmetric convex polygons or polyhedrons. In the 2D case the number V of vertices and number E of edges is 2N each and, if the grid vectors have rotational symmetry, the growth form is a regular 2N-gon with radius (N/2) tan(π/(2N)).

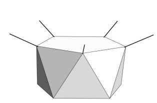
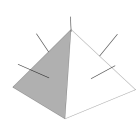
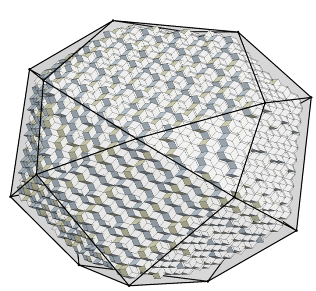
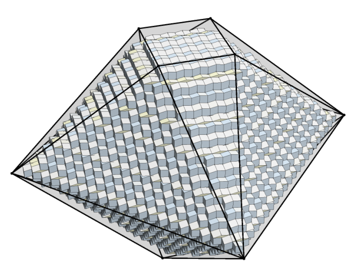
For the 3D case we define: A **r-tuple** is set of r grid vectors. A r-tuple is called **flat** if all its vectors belong to one plane. A flat r-tuple is called **complete** if none of the remaining N-r grid vectors belongs to the plane of the r-tuple. Let k(r) be the number of complete r-tuples. Then we have V, E, F (for faces):

$$V = N(N-1) - \sum_{r\geq 3} k(r)(r*r - r - 2), \qquad E = 2N(N-1) - \sum_{r\geq 3} 2k(r)(r*r - 2r)$$

$$F = N(N-1) + 2 - \sum_{r\geq 3} k(r)(r*r - 3r + 2)$$

If there are no flat r-tuples for r≥ 3, the sums do not apply. Each complete r-tuple produces a vertex of degree 2r, remaining vertices have degree 4. **Table 1** gives examples without and with complete 3- tuples.

Table 1: *Examples of growth forms with N = 5.*

Grid vectors		
Growth form		
r	-	3
k(r)	-	2
V, E, F	20, 40, 22	12, 28, 18

3D Ammann Tiling

Probably the best known 3D grid tiling is the Ammann tiling with icosahedral symmetry, **Figure 3**. Its 6 grid vectors are derived from the dodecahedron and the growth of the tiling was the theme in [5]. The form turned up as a surprise in computer experiments. Now it is clear that the visual impression is correct: The growth form is an Archimedian polyhedron, the icosidodecahedron inscribed in a sphere of radius $\sqrt{5-2\sqrt{5}}$ [4].

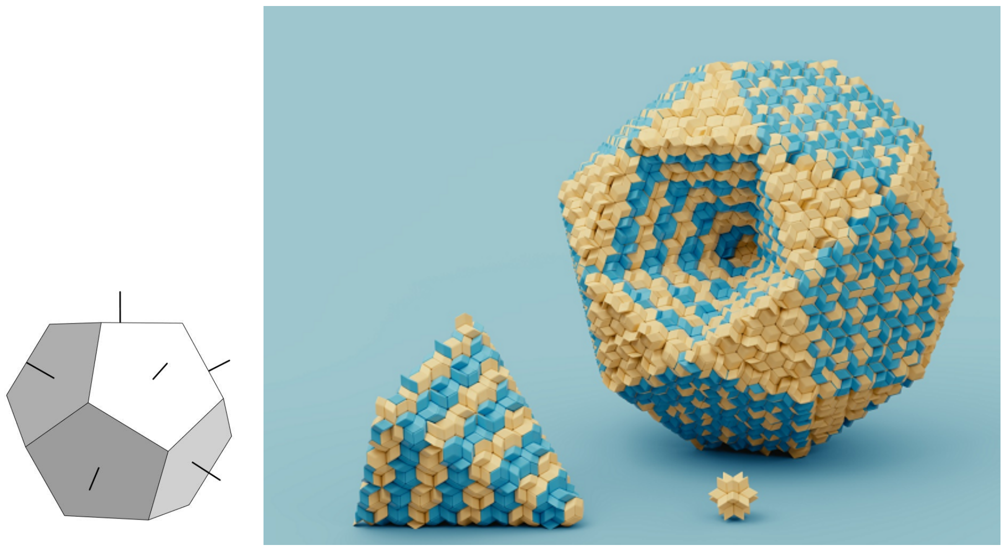


Figure 3: *Grid vectors and growth of the 3D Ammann tiling.*

Summary and Conclusions

We can explicitly calculate growth forms for any regular grid tiling in 2D and 3D. An interesting unsolved problem is to find full descriptions of all polyhedra which are growth forms of grid tilings.

References

[1] S. Akiyama and K. Imai. "Cellular Automata and Discrete Complex Systems." In *Lecture Notes in Computer Science, Volume 9664,* edited by David Hutchinson et al., Springer, 2016

[2] S. Akiyama, J. Caalim, K. Imai, H. Kaneko. "Corona Limits of Tilings: Periodic Case." *Discrete and Computational Geometry,* vol. 61, no. 3, 2019, pp. 626-652

[3] N.G. de Bruijn. "Algebraic theory of Penrose's non-periodic tilings of the plane." *Kon. Nederl. Akad. Wetensch. Proc. Ser. A*, vol 84, 1981, pp.35-48

[4] D. Demski, P. Hilgers, and A. Shutov. "Growth forms of grid tilings." *Acta Crystallographica A,* vol. 78, no. 4, 2022

[5] A. Gross and P. Hilgers. "Corona 27." *Bridges Stockholm 2018 Art Exhibition Catalog.* http://gallery.bridgesmathart.org/exhibitions/2018-bridges-conference/hilgers

[6] P. Hilgers. "Late arrival." *Bridges Aalto 2022 Art Exhibition Catalog.* http://gallery.bridgesmathart.org/exhibitions/2022-bridges-conference/hilgers

[7] A.V. Shutov and A.V. Maleev. "Study of Penrose Tiling Using Parameterization Method." *Crystallography Reports*, vol. 64, no. 3, 2019, pp. 351-361

[8] A.V. Shutov and A.V. Maleev. "Layer-by-Layer Growth of Ammann-Beenker Graph." *Crystallography Reports*, vol. 64, no. 6, 2019, pp. 851-856