Abelian Sandpile Quilting Blocks

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Abstract

I use traditional quilting blocks with horizontal and vertical mirror symmetry and/or 4-fold rotational symmetry as boundaries for the Abelian Sandpile Model. I further experiment with using shapes within the quilting blocks to seed different quantities of sand throughout the pile before allowing the sand to topple and stabilize. I observe that the symmetries of the underlying quilting block are maintained, and that the shape of the original seed can be found in the final stabilized pattern.

Introduction

The **Abelian Sandpile** is a mathematical model created to give an intuitive understanding of the ways natural systems (forests, snowy mountains, or piles of sand) can "self-organize" into a critical state. The model demonstrates that catastrophic events do not require dramatic external causes (like a lightning strike or a volcanic eruption) [1]. An avalanche can start by dropping a single grain of sand. While the sandpile is simple enough to be implemented with a few lines of code, the patterns created are complex and beautiful. My goal was to experiment with seeding and the boundary around the sandpile to make something unique.

The Sandpile Model

Abelian Sandpiles are 2D *cellular automata*, defined on a bounded or unbounded lattice, with each cell in the lattice representing a whole-number quantity of sand. For this paper we exclusively consider grid lattices; each cell has four neighbors corresponding to the four cardinal directions: north, south, east, and west. When sand is added to an individual cell, the quantity is checked against the *toppling limit* (the quantity of sand at which the cell *topples*). I use a toppling limit of 4 for the sandpiles in this paper, meaning that if there are 4 or more grains of sand in a single cell, the cell is toppled, and sand is spread equally to its four neighbors.

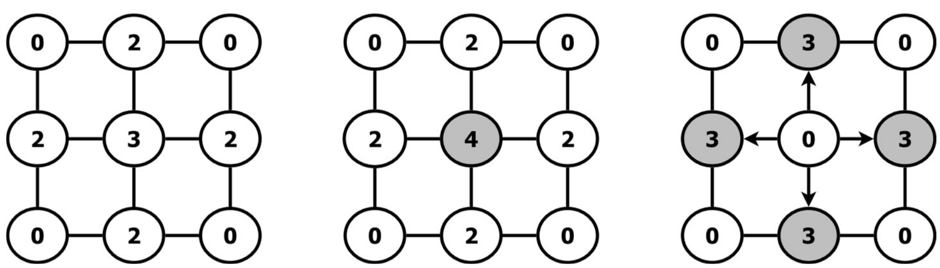


Figure 1: *Demonstration of toppling, with numbers indicating quantities of sand grains. Left: Initial state. Middle: 1 grain added to the center cell. Right: Center cell toppled, spreading 1 grain to each of its four neighbors.*

Although the model uses terms like *topples* and *avalanches* (many topples in a row), the imaginary sand does not behave like real sand. In Figure 1, sand is piled higher when the center topples, even though gravity would dictate that it should flow downward.

A classic example of the sandpile involves dropping a large quantity of sand on a single cell. That single stack spreads large amounts of sand to its four neighbors, which in turn spread sand to their neighbors, on and on through millions or even billions of topples. The system *stabilizes* when all cells are 3 grains of sand or shorter. The cell is then colored according to how many grains of sand remain in the cell: 0, 1, 2, or 3. The result is a fractal-like pattern (Figure 2).

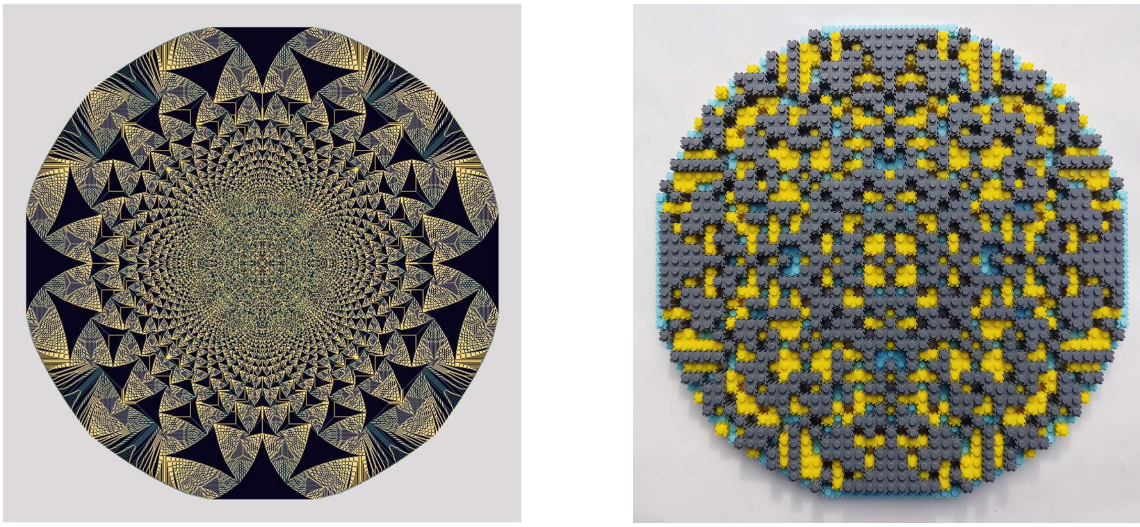


Figure 2: *Sandpiles with grains dropped on an infinite grid. Left: 1 million grains. Right: Sandpile with 3,450 grains, built in 3D using Pix Brix (an interlocking toy similar to LEGOs).*

For the sandpiles in this paper, I selected palettes from "Color Hunt" [4] and similar websites, picking color combinations intended to highlight details of the stabilized piles. The process was subjective, with the final choices based on what I considered aesthetically pleasing.

Bounding and Seeding the Sandpile

In an infinite grid, sand keeps spreading outward until it stops toppling. But in a bounded sandpile, we can imagine sand being dropped on a table, with some sand falling off the edge. The table's edge forms a boundary which we call the *sink*. Figure 3 shows a bounded 101×101 cell grid lattice with the sink represented by the square border.

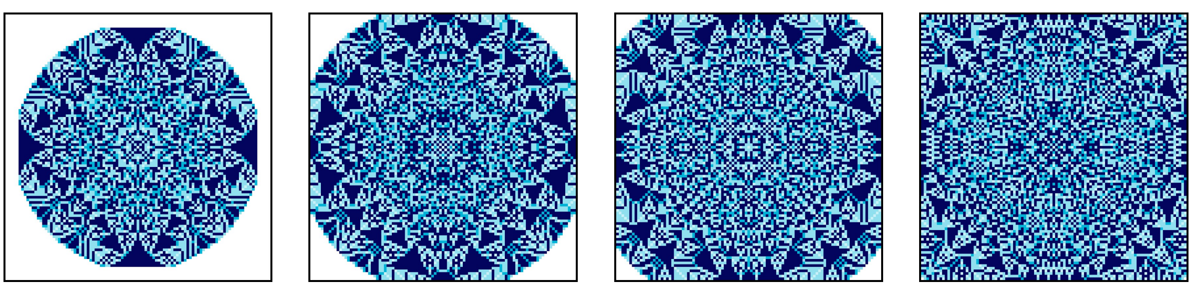


Figure 3: *All sandpiles above are bounded by a square sink, 101×101 cells. Left to right: 15,000 grains dropped in the center, 22,000 grains, 30,000 grains, 50,000 grains.*

But sinks are just one tool we can use with the sandpile model. We can also *seed* a sandpile by starting with grains already spread throughout the table [3]. In this way we can change the initial potential energy throughout the sandpile, measured by the quantity of grains in each cell [1]. While sinks place a boundary on sandpiles, seeding can change how they grow and the resulting stabilized shape.

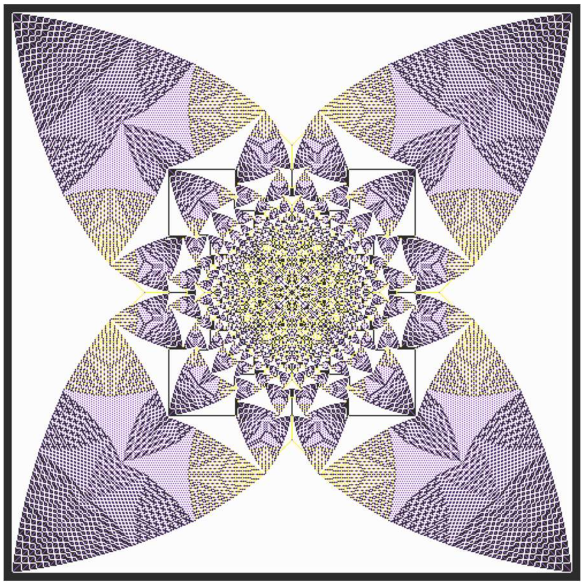


Figure 4: *The sandpile resulting from a uniform seed of 2 grains, with 100,000 grains added to the center.*

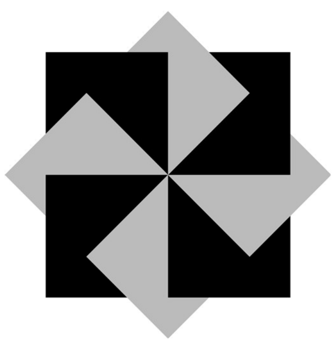
Figure 4 shows the stabilized pile after 100,000 grains are added to the center of an unbounded lattice uniformly seeded with 2 grains of sand. The resulting flower-like pattern and square border are not a sink, but rather the way the pile forms naturally. The white portions of the image represent cells that contain 3 grains of sand, while the dark gray border is the 2-grain seed just outside the natural spread of the pile.

With both seeds and sinks as tools, we can now affect the sandpile in new ways. We could continue to use uniform seeds and regular polygonal sinks, but I was curious how a more complex shape would behave, which led me to try quilting blocks.

Before landing on the final designs shown in this paper, I experimented with several other seeding techniques from Klivans, including negative seeds and alternating checkerboard seeds [3]. These informed my final seeding choice, which combines uniform and alternating seeds.

Quilting Blocks as Seeds and Sinks

Quilting blocks use simple shapes like triangles, squares, diamonds, and circles to make elaborate patterns. The blocks themselves are relatively small and can easily be stacked and stored until the final quilt is ready to be assembled (an essential feature when moving across the country in a covered wagon). Many patterns, such as "Double Card Tricks" [2] in Figure 5, can trace their names back to this frontier heritage.



4	2	2	2	2	4	4	4	4
4	4	2	2	2	4	4	4	2
4	4	4	2	2	4	4	2	2
4	4	4	4	2	4	2	2	2
2	2	2	2	4	2	2	2	2
2	2	2	4	2	4	4	4	4
2	2	4	4	2	2	4	4	4
2	4	4	4	2	2	2	4	4
4	4	4	4	2	2	2	2	4

Figure 5: *Left: "Double Card Tricks" quilting block [2]. Right: Seeds based on colors within the block.*

The quilting block gives us a unique way to seed the pile, seeding according to the internal colors and shapes. To avoid issues with rounding on a grid lattice, I chose quilting blocks in which all diagonal lines are at a 45-degree angle to the axes. My script used grayscale pixel matching to determine the seed value for each cell, so the source image needed to be the same dimensions as the final sandpile. The right image of Figure 5 shows the center of the block at the cell level. The point at the middle contains the tips of the 8 shapes within the block, so a good value for the seed might be the average of the seeds used for each color. However, I chose to simplify and pick one color over the other in these cases, as the source image itself was programmatically generated pixel-by-pixel to avoid anti-aliasing errors.

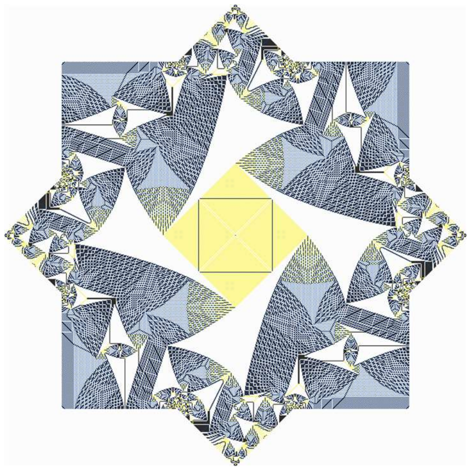


Figure 6: *"Double Card Tricks" sandpile.*

Figure 6 seeds the "Double Card Tricks" block with 2 grains of sand in the black sections and 4 grains of sand in the gray, on a 639×639 grid lattice. The gray sections are primed to topple, and we allow the system to topple and stabilize without adding additional sand to the center. The border of the quilting block acts as the sink, removing any sand that falls off the edge from the resulting pile.

"Double Card Tricks" has 4-fold rotational symmetry, as does the final stabilized pattern. Additionally, the seeded areas produce a pinwheel shape in the center, suggesting that the shape of the seed may be maintained in the stabilized pattern.

For the "God's Eye" block [2] in Figure 7, I seeded the black sections with 4 grains per cell and the gray sections with 2 grains per cell, on a 1,000×1,000 grid. This quilting block has horizontal and vertical mirror symmetry in addition to 4-fold rotational symmetry, and these properties are again maintained when we allow the pile to topple and stabilize.

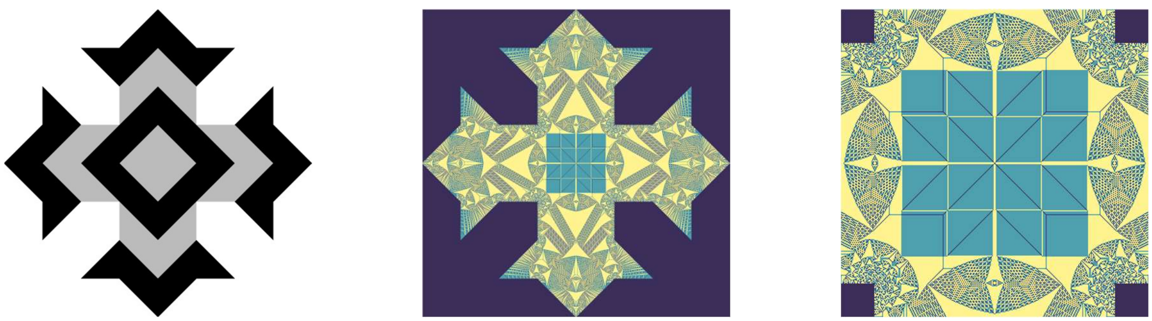


Figure 7: *Left: "God's Eye" quilting block [2]. Middle: Result after seeding, toppling, and stabilizing. Right: Zoom on center of the "God's Eye" sandpile.*

By happy accident, the center of the "God's Eye" pattern forms another common quilting block, an eight-point star [2], as well as formations that look like eyes. The gray parts of the original pattern can be seen in the long chains moving out from the center.

Summary and Conclusions

While not conclusive, initial experimentation does suggest that the shape of the initial seed is maintained in the stabilized sandpile. The degree to which the seed shape is maintained can be better quantified in future exploration. Also, quilting blocks with curved components, or internal shapes that do not translate well to a grid lattice, could be attempted with different lattices, such as triangular or hexagonal lattices.

References

[1] P. Bak. *How Nature Works: The Science of Self-Organized Criticality*. Springer-Verlag, 1996.

[2] M. M. Gordon. *1000 Great Quilt Blocks*. Collins & Brown Ltd., 2003.

[3] C. J. Klivans. "Chapter 5 – Pattern Formation." In *The Mathematics of Chip-Firing*. CRC Press, 2018.

[4] G. Shir. *Color Hunt*. 2022. https://colorhunt.co