Constructing Wooden Polyhedra

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Abstract

Methods for creating wooden polyhedra are discussed, leading the constructor and the viewer to think about 0, 1, 2, and 3-dimensional geometric components. The field has a long history in math, fine art, and education and carries symbolic importance for its tangible merger of beauty with structure and rationality.

Introduction

Wooden polyhedra can be beautiful objects: tactile, organic, naturally colored, and crisply faceted, with a lovely grained texture. They are solid, robust, and substantial compared to paper or cardboard models and show the craftsmanship of their individual creation. Holding them in the hand and turning them to see all sides can be a sensuous, artistic experience. The "viewer" is in fact a multisensory "experiencer," enjoying the texture, the weight, and the smell of the wood in addition to the visual appearance. They may serve as mathematical or artistic models of well known shapes already existing in the geometry literature, or they may be sculptures based on original forms that are created for aesthetic purposes. This paper surveys a variety of examples and construction methods and suggests that making wooden polyhedra has significant pedagogical value as a laboratory exercise in teaching mathematics. Included in the supplement [9] is a discussion of methods for students to produce wooden polyhedra in a school setting.

Well crafted polyhedra models that are presented with just the right amount of technical explanation can engage the viewer and lead them to further investigate the underlying mathematical ideas. Some museum collections contain wood polyhedra that historically were created for pedagogical purposes. An active current practice among hobbist woodworkers is to construct Platonic solids such as the dodecahedron and icosahedron as a demonstration of technical mastery [23]. In addition, in the fine arts world one finds interesting examples of significant sculpture that in physical terms are simply wooden polyhedra.

The study of polyhedra develops spatial reasoning and imagination while providing foundations for 3D design. By creating one's own polyhedra, one comes to grips with their essential character. Technical details arise of edge lengths, polygon angles, dihedral angles, structural regularities, transformations, etc. Many references recommend methods for students to learn geometric concepts and mathematical thinking via polyhedra, typically with paper (e.g., modular origami), cardboard (e.g., cut and glued models), or plastic construction sets (e.g., Polydron for face models or Zometool for models emphasizing vertices and edges). Making wooden models is easily within the skill set of typical hobbyist woodworkers.

Figure 1: *Polyhedra made with a wood panel for each face: (a) a 92-faced symmetrohedron, (b) the Minkowski sum of five tetrahedra, which has 80 faces, and (c)"jtD" in Conway notation, with 90 faces.*

Some Examples

Despite its long history, the term "polyhedron" is notorious for being vaguely and inconsistently defined. Depending on one's field of mathematics, polyhedra may be embedded 3D volumes, or merely their 2D surfaces (as a "torus" is the *surface* of a donut, not its volume), or something more abstract involving essential incidence relations of primitive elements. Definitions differ on conditions such as convexity, connectedness, finiteness, and self-intersection. We will ignore these fine points and assume an intuitive sense of the vernacular usage. This presentation is organized around how models emphasize the 0, 1, 2, or 3-dimensional elements of a polyhedron, i.e., the vertices, edges, faces, or solid content:

Face Models. Figure 1 shows three examples of face models, constructed with a separate piece of wood for each face to emphasize the 2D planar elements. Figure 1a shows a 92-sided symmetrohedron [15], notable because it features twenty regular enneagons (a.k.a. nonagons), a polygon that is relatively scarce in the geometry literature. It is assembled from polygonal panels of maple, bubinga, and purpleheart. Figure 1b shows the Minkowski sum of five regular tetrahedra [24]. Its faces are 20 equilateral triangles of black walnut and 60 rhombi of sapele. While the uniform compound of five regular tetrahedra is a very familiar structure and the Minkowski sum is a well studied operation, this wooden model is unique in combining these ideas. Figure 1c is a lovely 90-sided form with no particular name. It is a type of enneacontahedron succinctly called "jtD" in Conway's polyhedron notation. The 30 rhombi are maple and the sixty kites are bubinga. I constructed these three models around 2002 from quarter-inch thick panels by a method detailed below. Each is approximately 11 inches in diameter.

Volume Models. Moving up a dimension from surfaces to volumes, Figure 2 shows three 3-inch models emphasizing the 3D body of polyhedra. Each is cut from a cube of solid basswood. Figure 2a is a rhombic triacontahedron (the Catalan solid bounded by thirty golden rhombi). Figure 2b is a snub cube (the Archimedean solid with one square and four equilateral triangles at each of its 24 vertices.) Figure 2c is a "Sharpohedron," (described and cut as a solid wood model by Abraham Sharp around 1700). It is a circumscribed form of what is called "jtT" in Conway notation. I cut these using a custom miter-box method [10], suitable in each case because six of the faces are remnants of the starting cube's faces, so at each step, uncut portions of the original faces orient the partially cut block within the miter box.

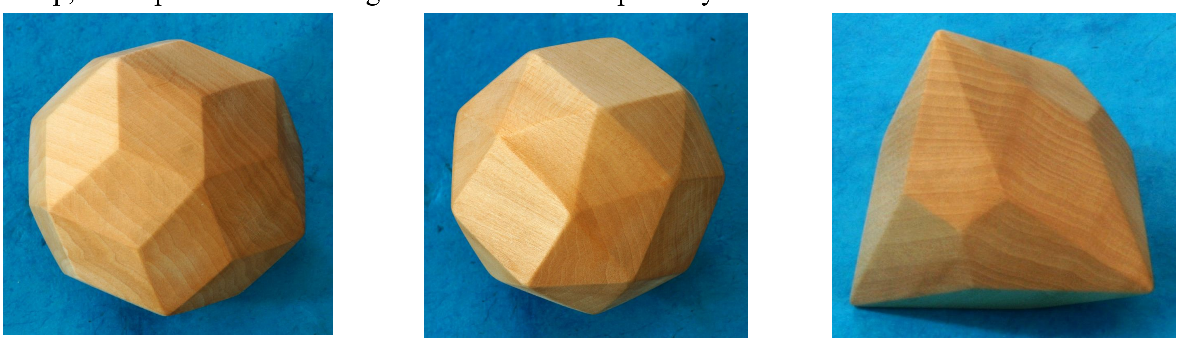

Figure 2: *Solid wood polyhedron models: (a) rhombic triacontahedron, (b) snub cube, (c) Sharpohedron.*

Edge Models. Edge models emphasize 1-dimensional components. Perhaps the most famous examples are the "skeletal" polyhedra in Luca Pacioli's 1509 book *Divina Proportione* [19]. Figure 3a shows an elevated icosidodecahedron from one of the surviving manuscript copies of the book (hand-written in 1498). Pacioli worked with Leonardo da Vinci for several years and credits him for the perspective polyhedra illustrations, but it is controversial whether any of the surviving images were hand-drawn by Leonardo or just copies by others of lost originals. (And there is no evidence about whether Leonardo was involved in constructing the wooden models on which the drawings are based.) Figure 3b shows my 16-inch diameter speculative reconstruction of what could have been the model for the drawing. The 120 equilateral triangles are each assembled from three mitered pieces of cherry wood, so 360 pieces had to be cut and assembled with precision joinery. The model in Figure 3b was displayed along with a 1509 printed edition of the book at the Oxford University Bodleian Library exhibition "Thinking 3D" [6]. Figure 4 shows another Pacioli example, discussed below.

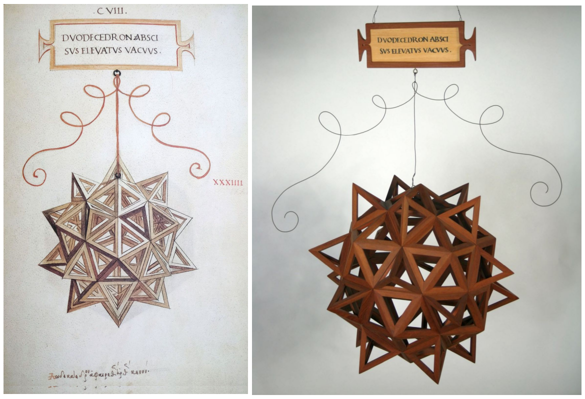

Figure 3: *Drawing of elevated icosidodecahedron from Pacioli and my speculative reconstruction.*

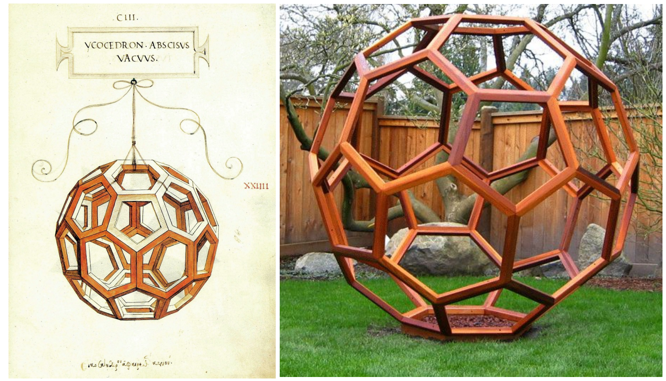

Figure 4: *Drawing of a truncated icosahedron from Pacioli and six-foot outdoor sculpture by Gary Moresky.*

Vertex Models. Vertex-focused models emphasize the 0-dimensional vertices of a polyhedron. For example, one can glue eight wooden spheres into a cubical structure. Or the natural way to hold four wooden bocce balls in one hand is to position them as the vertices of a regular tetrahedron. More interesting structures can be made by stringing wooden beads into a polyhedral form or by using other shapes than spheres for the units. Typical ball-and-stick chemistry models (e.g., cubane) can be seen as combined vertex and edge models. If the sticks are shortened to zero length, the balls abut to emphasize an arrangement of points in space. I have no interesting vertex models to present here, so I leave illustrations to the reader's imagination, but as a conceptual artwork, I'll suggest visualizing 27 wood cubes glued into a larger 3×3×3 cube, then seeing it as simultaneously a vertex, edge, face, and solid model.

A Sampling of Polyhedra in Fine Art

Pacioli traveled with wooden polyhedra models, using them didactically when he lectured. In the late 1400s, a set was purchased by the city of Florence for public display [20]. Although Pacioli associated with Leonardo da Vinci, his book was intended to teach math and was not directed to an art audience. A separate tradition of literature on art and perspective, also emphasizing polyhedral examples, was written for artists. Well known treatises on perspective by Piero della Francesca, Daniele Barbaro, Lorenzo Sirigatti, Jean Cousin, Jean-Francois Niceron, Albrecht Durer, Lorenz Stoer, Hans Lencker, Wentzel Jamnitzer and others each give a position of honor to interesting polyhedra [16, 22]. Precise geometric forms are powerful examples to show off one's mastery, as they require a clear understanding of the relationship between 3D objects and 2D projections. These works make clear that in a professional artist's practice, polyhedra are worthy objects of study, possessing a crisp beauty in their structural regularity.

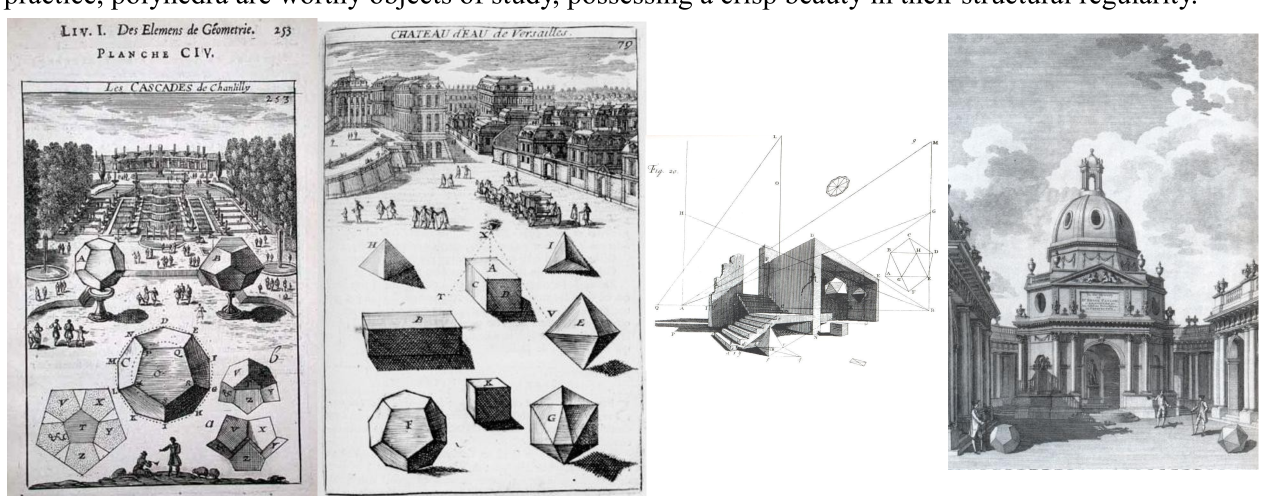

Figure 5: *Monumental polyhedral fantasies from Ozanam, Mallet, Taylor, and Malton.*

Figure 5 shows one plate each from Jacques Ozanam's 1693 *La Perspective Theorique Et Pratique*, Alain Manesson Mallet's 1702 *La Géométrie Pratique,* Brook Taylor's 1715 *Linear Perspective*, and Thomas Malton's 1775 *A Compleat Treatise on Perspective, in Theory and Practice*. What are those enigmatic giant polyhedra supposed to be? What materials would they be made of? How are humans expected to interact with them? Their large scale, relative to the architecture and formal landscaping, shows that the authors shared an aesthetic that gives conceptual importance to polyhedra in artistic thought. These monuments of the imagination appear to represent how essential values of structure, harmony, simplicity, and order connect mathematics with art. A similar conception appears to continue in much modern sculpture and a good many contemporary works are little more than polyhedral models.

I have seen many wooden reconstructions of Pacioli's famous models, sometimes in museum exhibitions celebrating Leonardo. (I am often surprised when the reconstructor's edge proportions in this context differ significantly from Leonardo's drawings.) In the mid-1990's I made a 16-inch diameter truncated icosahedron from cherry based on Figure 4a and proposed creating a complete set for an exhibition celebrating the book's 500th anniversary in 2009, but an appropriate venue never presented itself. I was, however, very pleased to see in 2003 a monumental construction by Gary Moresky, shown in Figure 4b. This impressive backyard sculpture is made from 2×2 cedar beams, cut, glued, and screwed together. It certainly embodies the monumental aspirations of the fantasies seen in Figure 5.

A few years later in 2006, Ai Weiwei hired craftsmen to construct for him a similar (9-foot diameter) truncated icosahedron edge model from huanghuali wood using traditional Japanese joinery. With Ai Weiwei's international fame as a sculptor and activist, it sold for almost a quarter million dollars and now can be seen at the Los Angeles County Museum of Art [1]. Interestingly, Weiwei reports he was not aware of the Leonardo/Pacioli drawings, but instead instructed his workmen to scale up a plastic cat-toy that was shaped like a truncated icosahedron. Close examination shows the beams lap on the interior in a manner distinct from the simple butt joint of all the other truncated icosahedron edge models I have seen.

The twentieth century sculptor Tony Smith is famous for a series of sculptures, *Moondog*, *Smoke*, *Smog*, and *Smug*, which are each large-scale assemblages of tetrahedra and octahedra. The tetrahedra are regular but the octahedra are stretched along a 3-fold axis. The units join on triangular faces to effectively make a faceted model of the diamond crystal lattice (with a tetrahedron at each carbon location and octahedra as connectors). Smith began by making large plywood models (painted black) on his lawn in Newark, NJ before having them fabricated in metal. The resulting forms have a surprising effect of appearing to tilt in different directions depending on one's viewpoint; it is amazing how this emerges from simple polyhedra.

The 20th century minimalists Sol Lewitt and Donald Judd each made many sculptural explorations based on cubes. Theirs are materialized, dissected, deconstructed, and agglomerated into larger structures in various ways, so there is much more to their intentions than just the cubic form. As Lewitt noted: “The most interesting characteristic of the cube is that it is relatively uninteresting. It is best used as a basic unit for any more elaborate function, the grammatical device from which the work may proceed.” Yet on a literal plane, apart from the conceptual ideas, many of these pieces are simply large wooden cubes.

Frank Stella's recent *Stars* series explores various riffs on the small stellated dodecahedron. Many involve formed materials with curved edges and surface elements, thereby distancing them from the strict geometry of the underlying polyhedron. But others with straight edges or planar faces are fundamentally large scale (5 to 12 foot diameter) wooden edge or face models of the small stellated dodecahedron.

In exploring the works of Olafur Eliasson, one encounters an enormous number of models and artworks inspired by polyhedral forms [4]. For example, his “The Missing Part Found” features a carefully mitered pear-wood edge model of a deltoidal icositetrahedron along with a thin metal-rod edge model of its dual rhombi-cuboctahedron, while “The Missing Part Reminded” similarly features a pear-wood tetrakis hexahedron edge model along with a thin metal-rod edge model of its dual truncated octahedron.

Let me conclude this polyhedral sampling with the work of the Japanese mathematician Tamekichi Hishida (1863-1943). I have learned little about his life except that he taught at the Tokyo Physics School and was also a tutor of Emperor Taisho. (He apparently came from an artistic family and was brother to the painter Hishida Shunsō.) I list him here with fine artists because he constructed the most astounding solid wood polyhedra I have ever seen. A collection of 47 models currently on display at the Museum of Modern Science of the Tokyo University of Science includes many nonconvex examples, such as the Kepler-Poinsot polyhedra, compounds of each Archimedean solid with its dual Catalan solid, and compounds of mirror-image enantiomorphs, among others [13]. Figure 6a shows the compound of the snub dodecahedron with its dual, the pentagonal hexecontahedron. This is the most difficult and intricate dual pair among the Archimedean/Catalan solids and it is a joy to see it so superbly executed. To hand-cut this chiral form so precisely from a solid block of wood requires not just astounding craftsmanship, but also the vision and motivation of an artist. Figure 6b shows an equally magnificent carving of the compound of the pentagonal hexecontahedron with its mirror image. I have never before in all my polyhedral explorations seen a physical model of this difficult compound. Figure 6c shows another unique example. It has no standard name but is formed of 60 kites and 90 rhombi and is called "jsD" in Conway notation. This is the convex hull of the compound shown in Figure 6a, so I assume it illustrates an intermediate stage in its carving process.

Hishida apparently hand-cut his models with a saw then carved the concave regions with a knife. Crisp lines are drawn to highlight the edges. He also made pedagogical models, showing the guide lines and initial cuts when starting from a cube to make a dodecahedron, icosahedron, or rhombic triacontahedron. They indicate he used a method akin to Abraham Sharp's 1717 methods [10], but Hishida went on to non-convex models, while Sharp limited himself to convex examples. I do not know whether Hishida's method derived from Sharp's book about making solid models or if he developed his ideas independently.

Figure 6: *Three hand-carved solid polyhedra models by Tamekichi Hishida.*

Construction Methods for Face Models

Studying polyhedra provides a gateway into an elegant world of mathematical beauty [3]. Constructing wooden polyhedra not only connects us with long cultural traditions in mathematics and the arts, but also immerses us in a world of precise geometric thinking. There isn't space here to give detailed instructions, but the following outline of methods explains the broad principles for general readers and should allow any experienced woodworker to fill in the production details.

Face models are easiest to begin with. First choose a model and determine its face shape and dihedral angles. If available, a laser cutter can cut accurate polygons, but the wood thickness is limited by the laser power. With a band saw or scroll saw one can quickly cut polygons of any shape and thickness, but the edges will not be precisely straight. In contrast, table saws and radial arm saws produce very straight edges, but have safety concerns. The solution I prefer (when not laser cutting) is to first cut roughly with a band saw, then use a disc sander (which is safer than any saw) to bevel and straighten the edges. First make an accurate face template (for each type of face), e.g., a regular pentagon of the desired edge length for a regular dodecahedron. Purists can make face templates by ruler and compass construction, but it is easier to use a drawing program and print on to card stock. For a longer-lasting template, laser cutting a thin piece of plywood or acrylic is excellent. Trace the template (using pencil) on to the wood an appropriate number of times and cut just outside the lines. Use a disc sander to sand exactly up to the line, tilting the table for half the desired dihedral angle. Test fit, brush glue on the sanded mating surfaces, and assemble the faces using masking tape to keep them in position while the glue dries. For large models, improvise some internal supports for added strength. When dry, remove the tape, clean up any errant glue, sand the surface as desired, and apply finish. The examples in Figure 1 were made by this method.

A good starting project is the regular dodecahedron. Its dihedral angle is 116.5 degrees, so each edge is beveled for half of that. Table gauges are set to read 0 when sanding 90 degrees to the wood surface, so the table tilt should be set to the complement of half the dihedral, i.e., 31.75 degrees. I have worked with high school shop teachers, teaching students how to make regular dodecahedra from plywood and found it succeeds as a very satisfying hands-on activity that ties nicely to the math curriculum. Many polyhedra can be made in this way, but it may come as a surprise that the simplest polyhedron—the regular tetrahedron—is the hardest. With a dihedral angle of 70.5 degrees, its edges are so sharp that the sander's table would need to be tilted beyond 45 degrees, which is not possible with common shop equipment.

The method can be adapted for non-convex polyhedra and compounds. For some impressive examples of large-scale non-convex polyhedra rendered as face models in painted wood, see Dale Seymour's garden size compound of five tetrahedra and his compound of five cubes, each over 4 feet in diameter [21]. (At that scale, a saw is more appropriate than sanding for making the bevels.)

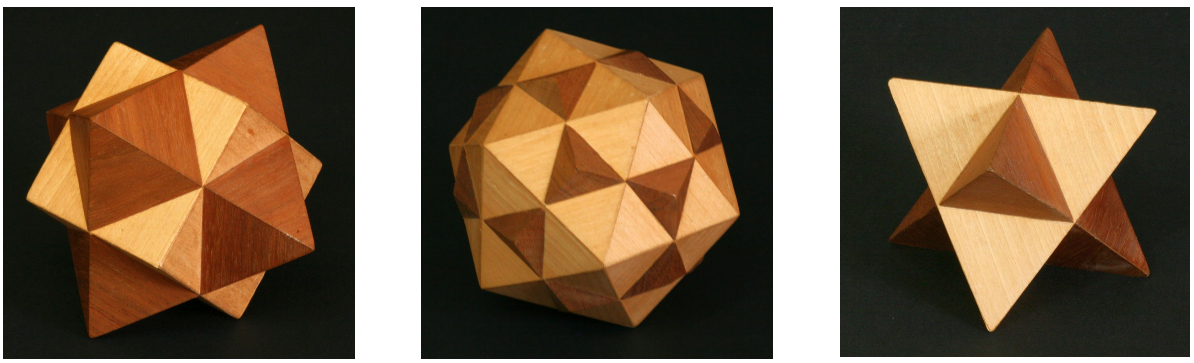

Figure 7: *Dual pairs: (a) cube and octahedron; (b) dodecahedron and icosahedron; (c) two tetrahedra.*

Figure 7 shows three non-convex models, about 6 inches in diameter, illustrating the Platonic solids in dual relationships. I made each of these models in two stages, first making a face model from maple as described above, then suggesting the interpenetrating dual polyhedron by adding a small pyramid of black walnut to each face. These are photos from when the models were 25 years old, after being passed around many classrooms and lecture halls, indicating how hardwood can be a very robust material for pedagogical models.

A Construction Method for Edge Models

I don't recommend edge models (Figures 3b and 4b) for beginners because of the precise joinery required to securely mate the struts on their short bevelled ends. Instead, my 2003 sculpture *Mother and Child* (Figure 8) illustrates an easier technique if one has access to a laser cutter. This ten-inch sculpture consists of two independent orbs, made from quarter-inch thick aspen. The inner one is a (2,1)-Goldberg polyhedron [8] face model. A laser-cutter was used to produce its 12 regular pentagons and 60 irregular hexagons. The outer orb is an edge model of a (3,1)-Goldberg polyhedron. Its 132 faces were each cut as a single hollow piece. There are twelve small pentagons and sixty each of two shapes of slightly irregular hexagons. A laser cutter can easily pierce an opening in a panel (unlike, say, a band saw), making this method easy for hollow faces. When sanding the bevels, I stopped just before the laser-cut edge, to leave a dark margin to highlight the edges. Each hollow face is a single unit, so this style can be seen as a blend of edge and face model; choosing the size of the openings can emphasize one interpretation or the other.

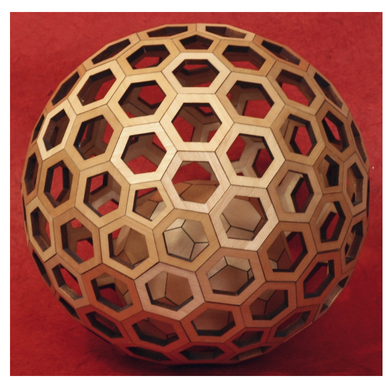

Figure 8: *"Mother and Child," edge model and face model.*

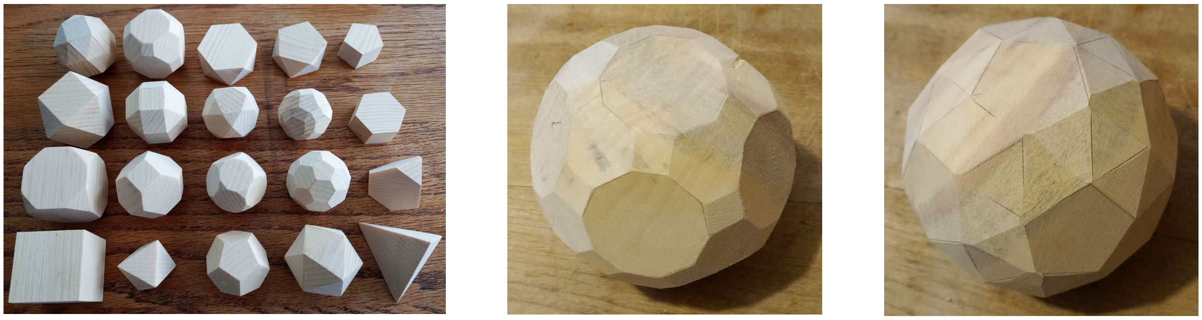

Figure 9: Solids: *(a) Platonic and Archimedean, (b) great rhombicosidodecahedron, (c) snub dodecahedron*

Construction Methods for Solid Models

Solid models require mastery of 3D geometry. A *truncated X* polyhedron can literally be made by sawing off corners from an *X,* but setting up a process to produce the proper angles and depths for the cuts requires mathematical care. Starting from a raw block of material, e.g., a cube, a series of steps must be made in which the slicing plane is accurately positioned relative to the surfaces of the partially cut block.

Mathematical models for education were produced in Germany in the nineteenth century and many survive in historical collections, including solid wood polyhedra, e.g., three Kepler-Poinsot polyhedra are on display at the University of Groeningen [7], but I have not found how they were produced. Three of Sharp's solid wood polyhedra from 1717 survive and are on museum display in Bradford: a rhombicuboctahedron, an icosidodecahedron, and a snub cube. His methods are well documented [10].

In the cabinet making and architectural woodworking traditions, there is a long history of making solid wood polyhedra to serve decoratively as finials, newel post tops, and similar ornaments. A comprehensive reference is by Holtzapffel, who in 1856 described the jigs, angles, and sequences of cuts to produce a wide variety of geometric solids with a table saw. He viewed it as a recreational activity and concluded: “a vast number of even the most complex models of geometrical and crystallographical solids, with plane surfaces, may be produced with comparative facility and great exactness, by the saw machine; and the mechanical amateur will find it a somewhat fascinating study” [14]. But like other texts I have seen in this cabinet-making literature, the naive methods he presented for cutting the dodecahedron and icosahedron center on a 5-fold axis and so miss the elegant method already detailed in 1717 by Abraham Sharp [10].

Euclid famously constructed the regular dodecahedron by adding six “roofs” to a cube. His method insightfully aligns three of the 2-fold axes of a dodecahedron with the three 4-fold axes of a cube. This amounts to centering six of the dodecahedron's edges in the six faces of a cube. After working out that there is a 31.7 degree angle between the cube's face and the dodecahedron's face, it is straightforward to set a saw to cut twelve times at this angle and remove everything from the cube that is not interior to the dodecahedron. This key insight was used by Sharp to cut not just the dodecahedron and icosahedron, but many other polyhedra with icosahedral symmetry. Once understood, the idea is so fundamental and natural that it has even been proposed as a kindergarten activity to extend Froebel's gifts in a 1906 book by Minnie Glidden [5], but instead of using a wood-saw, she cuts clay cubes with a knife. Nowadays, YouTube has many woodworking videos demonstrating similar methods [23].

Very precise wood polyhedra are often used as construction units in the fabrication of geometric puzzles. Stewart Coffin details how to make and use jigs and a sliding table to accurately cut truncated octahedra, rhombic dodecahedra, etc. with a table saw. [2] Similar setups, adapted for the appropriate cutting angles, can theoretically produce any desired polyhedra. Hiroshi Nakagawa has produced a pamphlet describing the methods he uses to make beautiful models, including all the Archimedean solids, from Japanese cypress wood [17]. (In some cases he first cuts temporary facets used only for alignment purposes.)

An experienced woodworker should find these materials to be a sufficient reference for setting up their saw to produce interesting solid polyhedra, but I can not recommend a table saw for novices because there are many safety issues. A significant number of woodworkers are walking around with fewer than ten fingers. Instead, I suggest that it is safer to begin by using a power disc sander than a saw. A sander removes material in a different manner than a saw: the work-piece moves into the plane of the disc instead of along the plane of a saw blade. So the process is slightly different, but the fundamental geometric knowledge required—the angles between planes and depth of material to remove—is unchanged.

Figure 9a shows Platonic and Archimedean solids produced in this manner from 1.5 inch cubes of pine. Some prisms and antiprisms are included, being in the same family with regular faces and equivalent vertices. Figures 9b and 9c show a great rhombicosidodecahedron and a snub dodecahedron also made by sanding, but from a larger 4-inch cube. The necessary angles and depths can be worked out from references in the geometry literature [3, 12, 18]. The supplement [9] to this paper gives technical details.

Conclusions

Polyhedra, with their timeless beauty, simplicity, and symmetry, are essential objects to the human imagination. Building wooden polyhedra connects us physically with enduring ideas that span the centuries, from Platonic formulations, through the Renaissance, touching abstract and minimalist twentieth century art, and on to the present. They may gain extra grandeur when elevated by large size, worthy materials, and/or masterful craftsmanship, but simply appreciating them in the mind's eye is the crucial start. One might eventually work systematically through categories such as the Platonic solids, Archimedean solids, Catalan solids, symmetrohedra, dual pairs, and compounds, etc., but I recommend you start by simply picking an inspiring form and doing whatever it takes to create it. In the process you will get to know its vertices, edges, faces, and solidity, and will also come to better appreciate the natural properties of wood: how it is pliant enough to form with common equipment, yet strong enough to hold its form for decades. An elegant wooden model sitting on your desk is an ageless embodiment of pure geometry. When you or a passerby picks it up, holds it, and spins it in the hand, the minimalist idea will manifest itself that reducing form to its essence is a way of avoiding distractions and seeing anew the beauty in simplicity. Enjoy the multifaceted experience!

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