Prime Factorization Fractal Tilings

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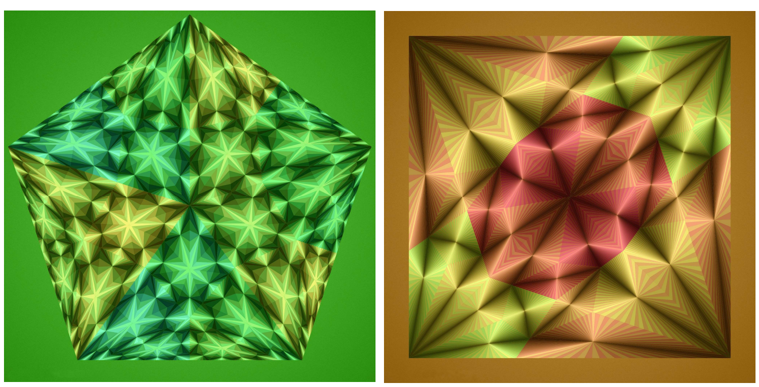
Abstract

This paper introduces an approach for creating tilings made by the iterative fragmentation of polygons according to the prime factors of any integer greater than 1 (n), thus attempting to link the properties of n to a tessellation made of n tiles. Choosing n as the power of a base integer, the method produces non-hyperbolic bounded fractal tilings, rich in visual complexity. In addition, the paper presents a colouring method, which iteratively applies the prime factorization fractal approach to the luminosity dimension, thus synchronising colour and geometry along the iterative tiling process.

Prime Factorization Tilings

As I am fascinated by Number Theory, in order to visualize number properties, I explored the idea of fragmenting regular polygons into tessellations according to the prime factors of an integer [2]. The proposed tiling approach starts from selecting an integer ($n \geq 2$), and decomposing it into its prime factors sorted for example in ascending order ($n = p_1\ p_2\ \ldots\ p_k$). Then it selects an initial regular polygon, whose number of sides is function of p_1. Afterwards the initial shape is fragmented into p_1 tiles, similarly to "splitting a pie". Each resulting tile is then furtherly fragmented into p_2 tiles, and the process is iterated until the prime factors of n are exhausted: the substitution rule at level i is determined by p_i. The fundamental theorem of arithmetic ensures that each n produces a unique tiling structure, made of n tiles (Figure 1).

In this paper I extend my idea by investigating the case where n is the power of an integer base ($n=b^l$) and produces a fractal as the exponent (l) increases. In order to reach this objective, I adopt a new approach for the iterative tiles splitting, suited to fractals; and I introduce a colouring method, which consists of a one-dimension prime factorization fractal synchronised with the geometry.



(a) (b)

Figure 1: *Prime factorization of (a) $n=4025=5\cdot5\cdot7\cdot23$ and (b) $n=4068=2\cdot2\cdot3\cdot3\cdot113$, from[2].*

The Geometry of Prime Factorization Tilings

In order to split a polygon (tile $\boldsymbol{T_j}$, where j is the *tile index*) with e edges into p_i sub-tiles, I apply four steps:

A) Select a central point within the polygon (*central vertex* $\boldsymbol{V_c}$), e.g. the barycentre

B) Define a reference point on the tile border (*synch point* $\boldsymbol{V_0}$). In this paper I choose the vertex V_c of the *parent tile* T_p (tile from which T_j was originated), i.e. $V_0(T_j)=V_c(T_p)$

C) Create the sub-tiles by splitting the border in p_i equal parts (*split metric*), starting from V_0 and connecting each two selected consecutive vertices to V_c. In some variations the starting point can be taken at a defined distance from V_0 (*shift* $\boldsymbol{S_0}$) producing alternative interesting configurations.

D) When necessary, normalize the resulting tiles order by appropriately adding vertices on the edges (*tile normalization*).

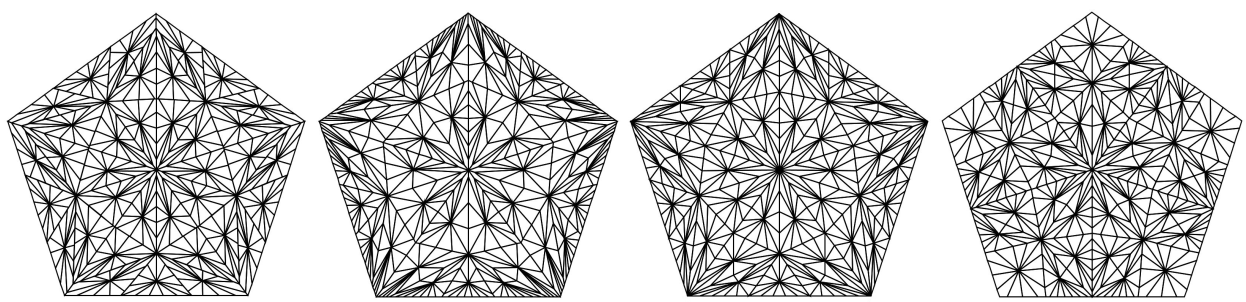
The above steps A-D are jointly defined in order to reach two objectives that contribute to the final symmetry of the image: (*Ob1*) adjacent tiles are split so that their sub-tiles are aligned on vertices' positions along the common border, (*Ob2*) the produced sub-tiles have a coherent vertex numeration, therefore enabling a coherent splitting and colouring in the following iteration phases.

The driving choice is the definition of the split metric (C). A natural method would be to divide the tile perimeter in equal p_i parts using Euclidean metric ($\boldsymbol{E(P_1, P_2)}$ in the following), however this approach does not align the sub-tiles vertices during the iteration process (*Ob1*), see Figure 2 (a) for n=455. In [2] I introduced a specific metric, defining the distance of two points P_1, P_2 ($\boldsymbol{GE(P_1, P_2)}$) as:

$GE(P_1, P_2) = E(P_1, P_2)/E(V_{i-1}, V_i)$, if P_1 and P_2 are on the same edge delimited by V_{i-1}, V_i

$GE(P_1, P_2) = GE(P_1, V_i) + k + GE(V_{i+k}, P_2)$, if P_1 and P_2 are separated by vertices $V_i, V_{i+1} \ldots V_{i+k}$.

The proposed metric *GE* combines Graph and Euclidean distances, and requires an ordering of the vertices, which is carried out in step (B). This approach has to be used together with the normalization of all sub-tiles to order 4, which is done for triangles by adding a vertex V such that $GE(V_0, V)=1.5$, that is by adding a vertex at the midpoint of the external edge vs V_0. The final result is shown in Figure 2 (b) for n=435. Interesting patterns are obtained also by using a shift $S_0=e/2$, which preserves the overall symmetry.



(a) (b) (c) (d)

Figure 2: *Tiling of $n=455=5\cdot7\cdot13$ by approach: (a) Euclidean, (b) Graph-Euclidean, (c) Modular-3, (d) Modular-4.*

A further approach is to split a tile with e edges in p_i sub-tiles by assigning to each edge p_i/e (quotient) sub-tiles, and distributing the remaining p_i *modulo* e (remainder) sub-tiles appropriately in order to keep the symmetry. For example, for e =3 and p_i =5 by assigning the remainder 2 tiles, one to the edge V_0-V_1 and one to the edge V_2-V_0, and for e =3 and p_i =7 by assigning the remainder 1 tile to the edge V_1-V_2.

I investigated two alternatives: 1) "*Modular-3*" in which all created tiles are triangles, achieved by ensuring that at each step sub-tiling always includes the vertices of the *parent tile* (Figure 2-c); 2) "*Modular-4*" in which any created tiles has order 4, achieved by normalizing triangles into quadrilaterals, e.g. by adding an additional vertex at point P: $GE(Vo, P)=1.5$ (Figure 2-d). Modular-4 approach is employed by adopting a shift $S_0=0.5$ (in GE metric). In both approaches a specific splitting choice is required for $p_i=2$.

Prime Factorization Fractals Tilings

Choosing $\boldsymbol{n=b^l}$, where $b=p_1 p_2 \ldots p_k$, and sorting the factors of n as ($p_1 \ldots p_k$, $p_1 \ldots p_k$, …) "*power-ordering*", the tiling process produces a fractal as the integer l (levels) increases; indeed, after that the initial polygon is split according to the factors of b (first level, $l=1$) each resulting tile will be further split according to the factors of b^{l-1}, producing a self-similar non-hyperbolic bounded tiling [1] (Figure 3).

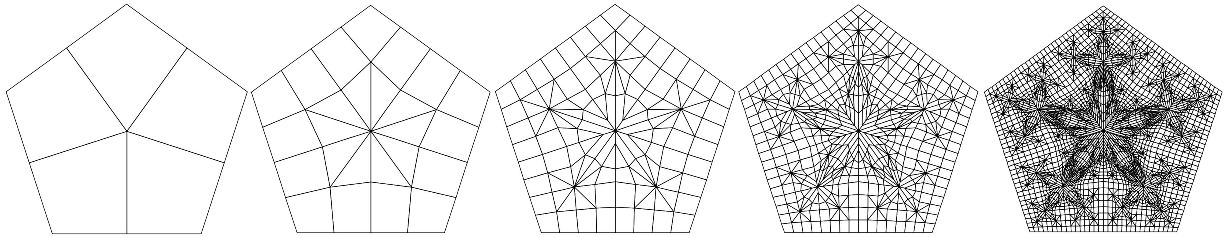
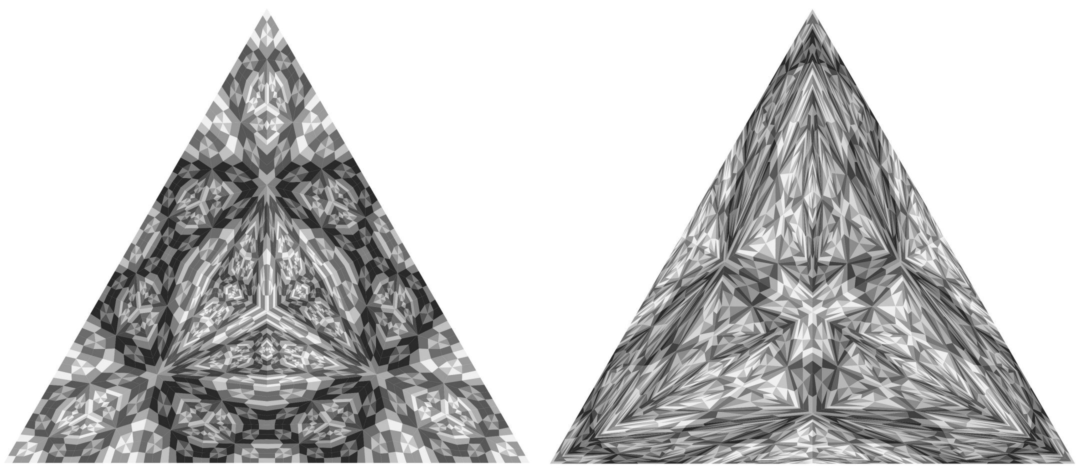


Figure 3: *Tiling of $n=5^1$, 5^2, 5^3, 5^4, 5^5, Modular-4 approach.*

The *Modular* approach, introduced in this paper, produces rich patterns for many *b* fractals, see Figure 4 for $n=15^3=(3\cdot5)^3$ created with *modular-4* and *Graph-Euclidean* approaches all other things being equal (using Binomial colour Intensity distribution, see further explanations).

The subcase $n=3^l$ in *Modular-3* approach, with a shift $S_0= 0.5$ and eliminating the central vertex, e.g. directly connecting the new vertices during the splitting process, is equivalent to the Sierpinski triangle.



(a) (b)

Figure 4: *Tiling of $n= (3\cdot5)^3$ by approach: (a) Modular-4, (b) Graph-Euclidean.*

From "Colouring the Fractal" to "Fractalizing the Colour"

In the previous artwork [2], based on sorting the factors p_i in ascending-order, I coloured the tiling in the Hue, Saturation and Intensity colour space (***H, S, I***) by defining *H* and *I* as functions of the *tile index* modulo the product of a subset of the prime factors of *n*. I selected larger factors for *I* and smaller ones for *H*: this approach puts in evidence the details through the luminosity, while providing an overview of the tessellation distribution through the Hue (Figure 1). *S* is defined as a linear function of *I*.

In the case of fractals, where a power-ordering is adopted, a different approach is necessary in order to maintain the structure self-similar while increasing *l*. I propose to produce a 1-dimension fractal of *I,* synchronised with the geometric splitting process. I assign iteratively the intensity to the tiles: given $n=b^l$, once defined the intensity I_a of a tile at the splitting iteration level *k* ($k=1\ldots l$), the intensities of the corresponding sub-tiles I_x ($x=1\ldots b$) at level $k+1$ are defined such as $average(I_x) = I_a$. This approach preserves the Intensity self-similarity through the iteration levels (Figure 5). I explored different distributions for the I_x values: e.g., linear, circular and binomial, in the last case for odd *b* assigning the average I_a to one tile. The distribution range used at each step has to be approximately normalised in order not to exceed the overall definition range of *I [0, 1]*, e.g. using a multiple of the distribution standard deviation calculated over l levels. Finally, different distributions can be mixed by a weighted sum for interesting artistic effects.

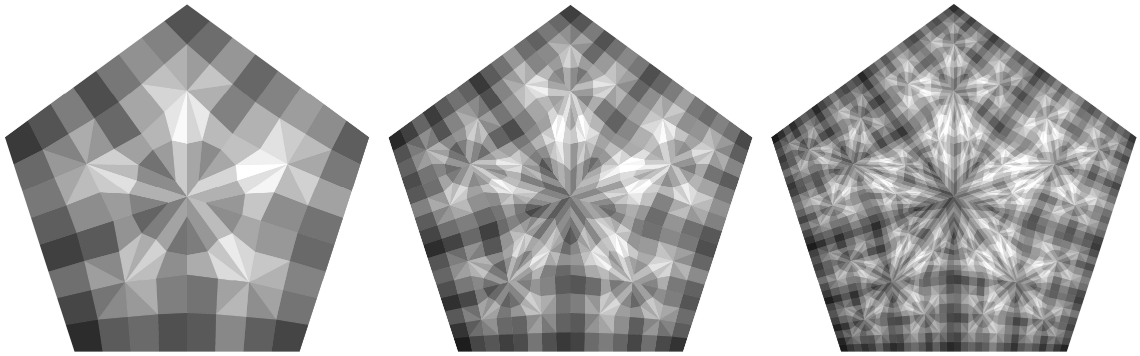
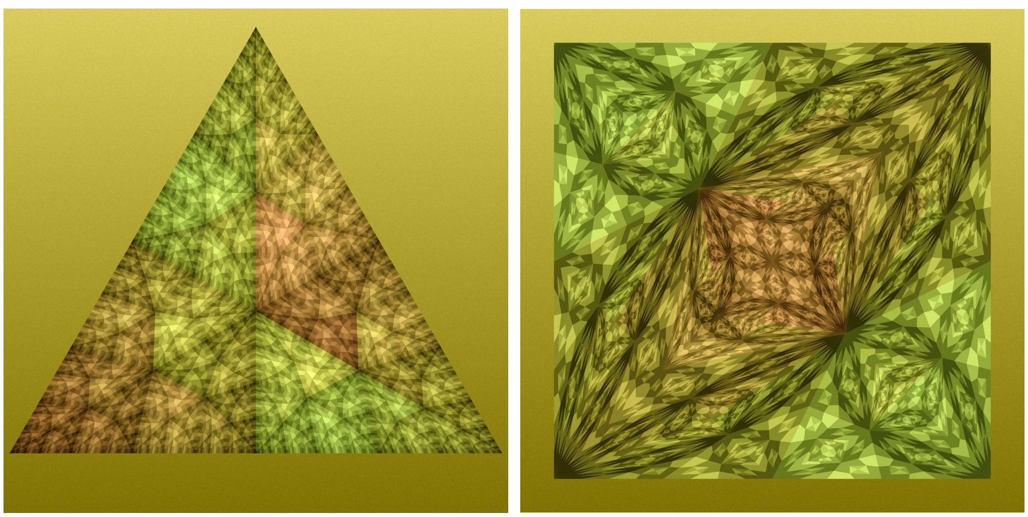


Figure 5: *Tiling of n= 5^3, 5^4, 5^5, Circular Intensity distribution, Modular-4 approach.*

For the proposed fractals I maintain the same definition of S and H used for the ascending ordering [2], as it has a consistent behaviour for fractals (Figure 6).



(a) (b)

Figure 6: *Tiling of n= (a) 3^8, (b) 10^4, Binomial Intensity distribution, Modular-4 approach.*

Conclusions

In this paper I attempted to build a *bridge* between the "hidden" *beauty of Number Theory* and the "evident" *elegance of Tilings* through fractal tessellations based on the structure of the integers' prime factorization, which acts both on geometry and colours dimensions. Several combinations of the described sub approaches can be used, resulting in different families of infinite tilings; I prefer some parameter sets depending on the used integer base. There are additional possibilities to be investigated, e.g.: fractalize also the Hue dimension, explore different ways for tiles normalization e.g. to order 4, represent integers' properties through tiles colouring e.g. totient function.

References

[1] R. Fathauer. "Self-similar Tilings Based on Prototiles Constructed from Segments of Regular Polygons". *Bridges Conference Proceedings,* Winfield Kansas, USA, July 28–31, 2000, pp. 285-292.

[2] S. Leonardo. "Integer factorization tesselations". *Proceeding of XXIV Generatve Art International Conference*, Cagliari, Italy, Dec. 15-17, 2021, pp. 473-476.