Artistic Depiction of Numbers Defined by Sets

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Abstract

Natural numbers can be defined using sets where zero is represented by the empty set and larger integers are defined recursively with the use of a successor function. Several original artworks are presented to illustrate this concept.

Mathematical Background

Number is a fundamental concept in mathematics, and there is a long history of humanity's collective interest in representations of quantity. Archaeological evidence indicates prehistoric people, circa 30,000 BCE, used a simple unary representation of tally marks on bones for counting [2, pp. 1–5]. As theoretical mathematics developed in the late 19th century and early 20th century, mathematicians began to explore more fully the nature of number and arithmetic [2, Ch. 12].

The development of set theory along with advances in logic created a framework for much of the axiomatic structure of the mathematics we use today. In 1925, John von Neumann presented a way of constructing the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ using sets [5]. In this set-based representation, zero is represented by the empty set: $\{\}$, which can in turn be represented by the numeral 0. Starting with $0 = \{\}$, we can define the natural numbers recursively with the use of a successor function, $s(n)$. The number $n + 1$, defined to be the successor of n, is the union of the set n with the set containing n, as follows:

$$s(n) = n + 1 = n \cup \{n\},$$

where n is the numeral representing the nth natural number. This recursive property of the natural numbers is commonly used by dictionaries to define small numerals beyond zero and one; for example three is defined as the "cardinal numeral next after two" [1].

Applying the recursive definition for the first few natural numbers is helpful in understanding von Neumann's construction. The successor of 0, $s(0)$, is 1:

$$1 = s(0) = 0 \cup \{0\} = \{\} \cup \{\{\}\} = \{\{\}\}.$$

The successor of 1, $s(1)$, is 2:

$$2 = s(1) = 1 \cup \{1\} = \{\{\}\} \cup \{\{\{\}\}\} = \{\{\}, \{\{\}\}\}.$$

The successor of 2, $s(2)$, is 3:

$$3 = s(2) = 2 \cup \{2\} = \{\{\}, \{\{\}\}\} \cup \{\{\{\}, \{\{\}\}\}\} = \{\{\},\ \{\{\}\},\ \{\{\}, \{\{\}\}\}\}.$$

The successor of 3, $s(3)$, is 4:

$$\begin{aligned} 4 &= s(3) = 3 \cup \{3\} \\ &= \{\{\},\ \{\{\}\},\ \{\{\}, \{\{\}\}\}\} \cup \{\{\{\},\ \{\{\}\},\ \{\{\}, \{\{\}\}\}\}\} \\ &= \{\{\},\ \ \{\{\}\},\ \ \{\{\}, \{\{\}\}\},\ \ \{\{\},\ \{\{\}\},\ \{\{\}, \{\{\}\}\}\}\}. \end{aligned}$$

In general, the number n is the successor of $n-1$ and is the set containing the numbers 0 through $n-1$:

$$\begin{aligned} s(n-1) &= (n-1) \cup \{n-1\} \\ &= \underbrace{(n-2) \cup \{n-2\}}_{n-1} \cup \{n-1\} \\ &= \underbrace{(n-3) \cup \{n-3\}}_{n-2} \cup \{n-2\} \cup \{n-1\} \\ &\vdots \\ &= \underbrace{0 \cup \{0\}}_{1} \cup \{1\} \cup \{2\} \cup \cdots \cup \{n-1\} \\ &= \{\} \cup \{0, 1, 2, \ldots, n-1\}. \\ &= \{0, 1, 2, \ldots, n-1\}. \end{aligned}$$

In von Neumann's representation of a given natural number n, the number of elements of that set (its cardinality) is just the number itself. For example, the set representing 8 is $\{0, 1, 2, 3, 4, 5, 6, 7\}$, which has 8 elements. Note these elements are generally sets containing other elements that are sets (that might contain other sets, and so on).

This von Neumann construction is far from concise, with many symbols used to represent even small natural numbers. Note that the commas are not needed, with juxtaposition being sufficient. For example, the number 3 can be represented as $\{\{\}\{\{\}\}\{\{\}\{\{\}\}\}\}$, requiring 16 symbols, even with the commas removed. The number of empty sets is 1 if $n = 0$ or $n = 1$, otherwise the number of empty sets is given by 2^{n-1}. The number of **{** symbols needed to represent n is 2^n. Similarly, 2^n **}** symbols are needed, resulting in a total of 2^{n+1} required symbols.

The point of this representation of numbers isn't efficiency of symbols, it is that a number can be defined in terms of the more basic abstract notion of a set. Thus having a robust theory of sets provides a solid foundation on the concept of number.

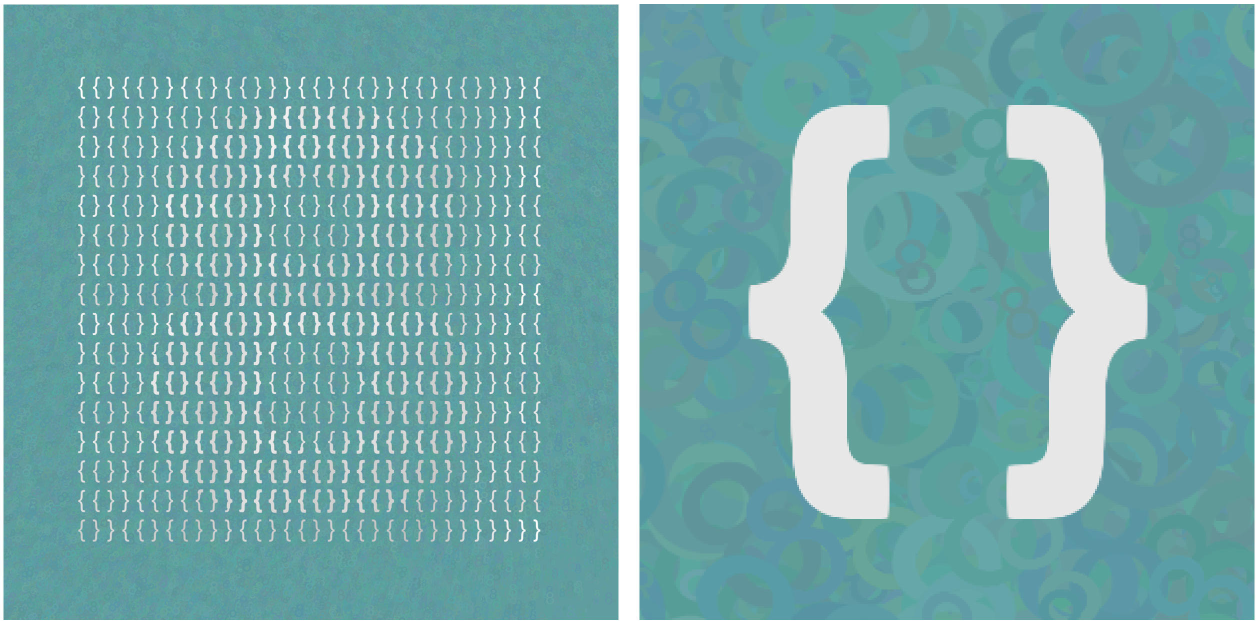
Original Artwork

Admittedly, the mathematical description of the von Neumann construction is rather technical and esoteric. In an effort to help explain this concept, I created several artworks.

The artwork *Braces of Eight*, shown in Figure 1, uses just the symbols **{** and **}** to represent 8, where their relative darkness corresponds to nesting depth. A total of $512 = 2^8 + 2^8$ symbols are needed to represent 8. The thickness of some symbols was varied to subtly show the numeral 8, shown in Figure 1a. The background texture is made from randomly sized, colored, and placed 8's, more easily seen in the detail image Figure 1b.

The artwork *Ordinal Eight*, shown in Figure 2, also depicts the number 8. Rather than the symbols **{** and **}**, the intuitive notion of a set being a container is used. In this artwork, the empty set is represented by a black square and larger sets are depicted by rectangles. Set membership is represented by nesting a rectangle inside an enclosing rectangle. The larger outermost rectangle contains eight smaller rectangles. It is instructive to compare the nesting structure of this piece with that seen in *Braces of Eight* (Figure 1).

A variant of *Ordinal Eight*, depicting the number 5, was created using LEGO™ and is shown in Figure 3. Blue 2×2 tiles represent the empty set. Set membership is represented by stacking layers on larger layers, where each layer is made from different colored plate pieces, as shown in Figure 3b.



(a) **(b)**

Figure 1: *The artwork* Braces of Eight *(digital print) shows the number eight represented as sets using just the symbols* **{** *and* **}**. *The full artwork is shown in (a) with detail in (b).*

Conclusion

The pieces described above use simple repeating elements to engage the viewer in the meaning of number and unique ways numbers can be defined and visualized. The work *Ordinal Eight* showing nested sets as nested regions has the same spirit as the *Von-Neumann Ordinals Series* by Jennifer E. Padilla [3] and the work *The von Neumann natural numbers* by David Pierce [4].

References

[1] "three, adj. and n." *OED Online*, December 2021.

[2] D. Burton. *The History of Mathematics: An Introduction*. McGraw-Hill, 2007.

[3] J. E. Padilla. "The Von-Neumann Ordinals Series." http://www.jenniferepadilla.com/detail/Von_Neumann_detail.html (accessed February 19, 2022).

[4] D. Pierce. "The von Neumann natural numbers." https://polytropy.com/2013/04/10/the-von-neumann-natural-numbers-a-fractal-like-image/ (accessed February 19, 2022).

[5] J. von Neumann. "An Axiomatization of Set Theory." *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. J. van Heijenoort, Ed. Harvard University Press, 1967. pp. 393–413. Originaly published in 1925.

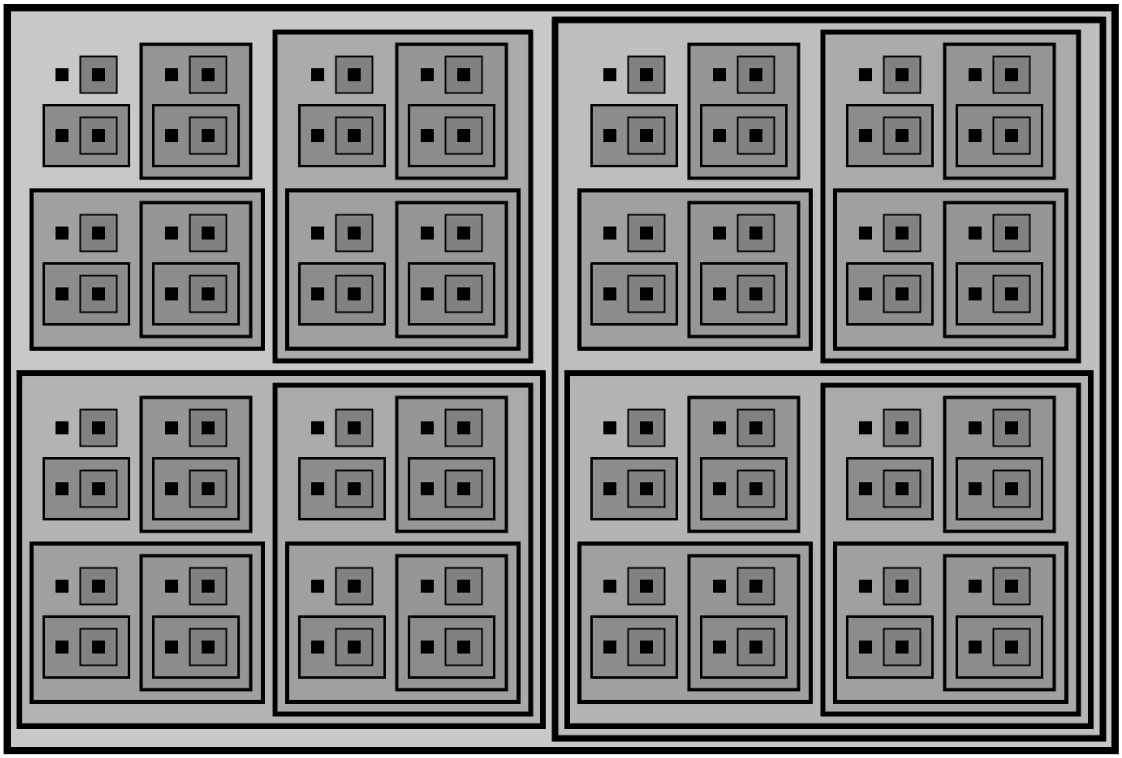
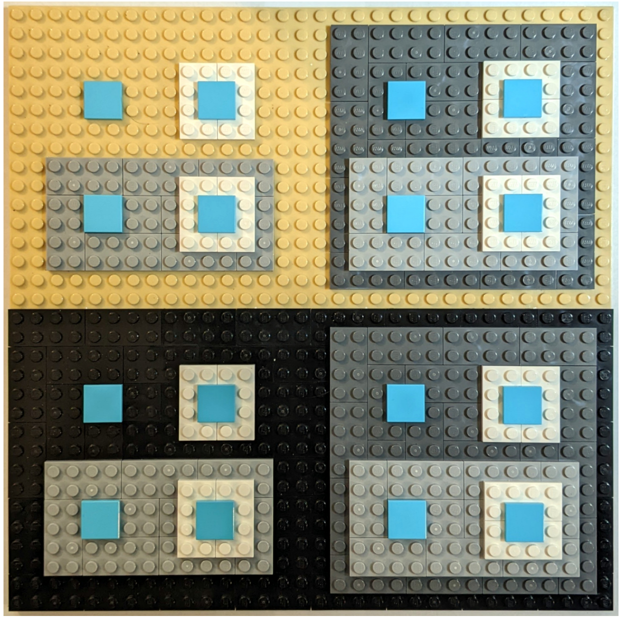
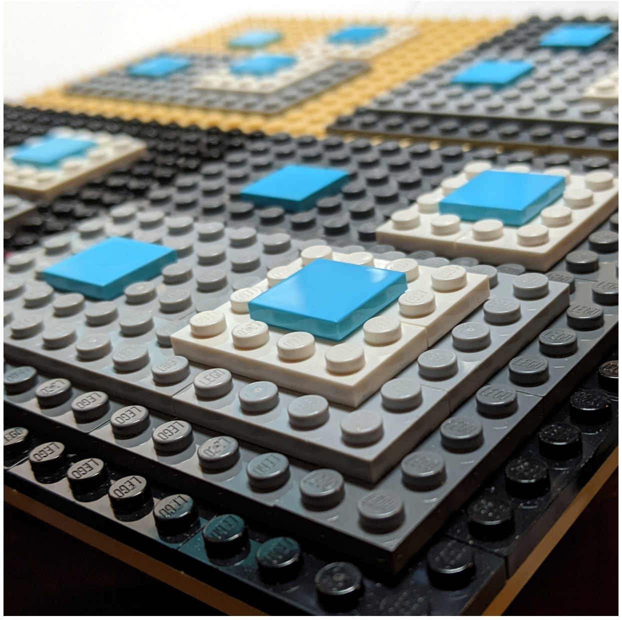


Figure 2: *The artwork* Ordinal Eight *(digital print) depicts the number eight represented as sets.*



(a)



(b)

Figure 3: *A variant of* Ordinal Eight *(see Figure 2), depicting the number five created using* LEGO™. *The full artwork is shown in (a) with detail of the stacked layers in (b).*