Generating Families of Islamic Star Rosette Patterns Based on k-Uniform Tilings

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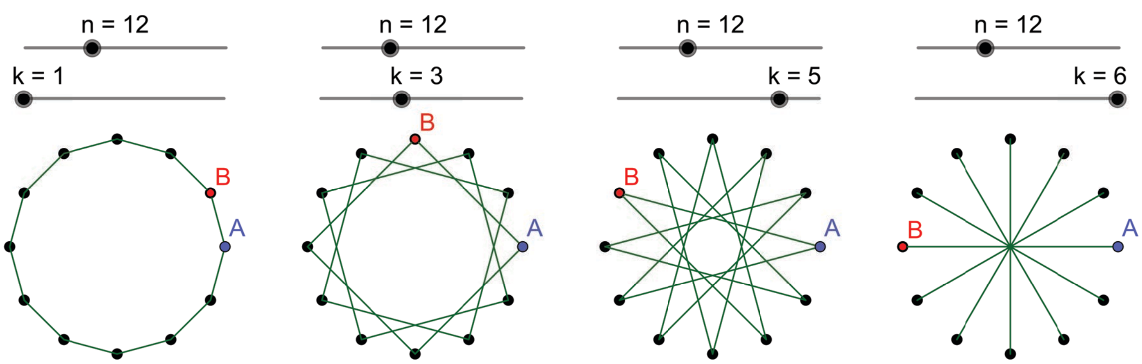
Abstract

Islamic star rosette patterns of a variety of symmetries are built on polygons and tangent circles in their underlying Euclidean compass and straightedge constructions. We present an algorithm for building families of star rosette patterns by situating the star polygons of these patterns inside the circles of a univalent packing whose intersection graph is any k-uniform tiling, and allowing for continuous variation of star polygon angle at points of tangency.

Defining Rosettes: The Star Polygon Does the Work

Star rosette patterns are found throughout Islamic art and architecture and are of interest to the authors due to the arrangement of tangent circles in their underlying structure. The foundation of an Islamic star rosette pattern is the **star polygon**. An $\{n/k\}$ star polygon is constructed on n vertices by drawing line segments connecting each vertex to the vertex k vertices away from it. When $k = 1$, you get a regular n-gon, and if n is even and $k = n/2$, you get an asterisk.



(a) **(b)** **(c)** **(d)**

Figure 1: *Examples of star polygons where* $n = 12$

A.J. Lee presents a detailed analysis of characteristics of star rosette patterns both simple and complex in [3]. Every crossing vertex should be degree 4, with equal opposite angles. That is, lines do not change direction at crossings. The hexagonal **petals** (highlighted in green in Figure 2) of the rosette that surround the central star polygon should have their outer four sides all equal in length. There is at most one k-value in the star polygon that allows for *parallel* petal sides. Smaller k-values will result in *convergent* petals that narrow as they extend outward, and larger k-values, up to $n/2$, will result in *divergent* petals that widen as they extend outward. A star rosette will typically have only one of a handful of possible angles, often the only distinction seen in some of the simplest patterns in different parts of the world, determined by the number of petals or divisions of the circle of the main rosette motif. Our construction approach allows for continuous variation in angle while still following traditional rules.

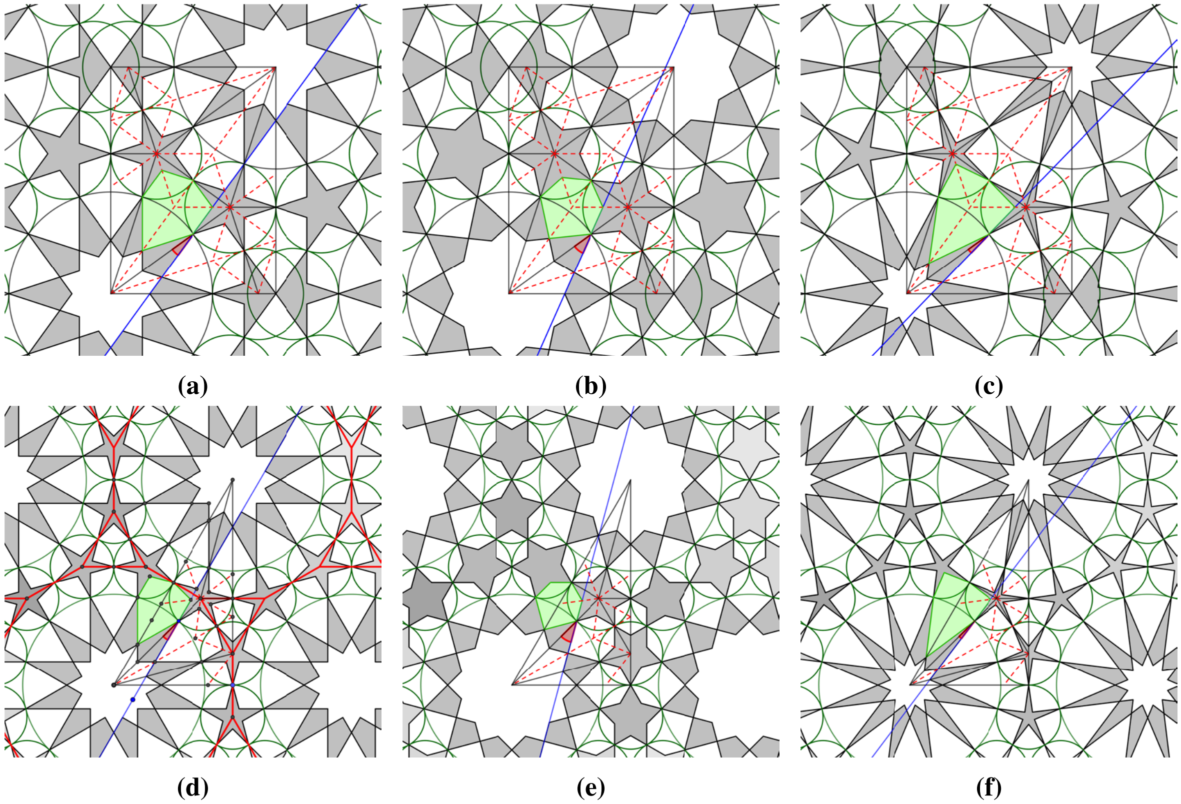


Figure 2: *10-fold (a-c) and 12-fold (d-f) star rosette patterns, with parallel (a,d), convergent (b,e), and divergent (c,f) petals highlighted in green. Note the overlapping circles necessary to make the 10-fold rosette pattern fit into a repeatable tile. 2d also includes its* [3.12.12] *uniform tiling.*

Regular Structure: K-Uniform Tilings and Circle Packings

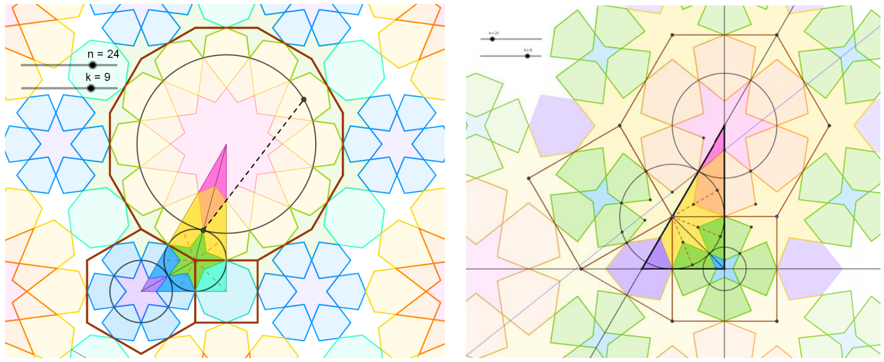
A k**-uniform tiling** is a repeating regular polygonal pattern that contains k different **vertex types**. Two vertices of a tiling can only share a vertex type if they also share congruent surrounding polygons and a **vertex transitivity class**, meaning that one vertex can undergo a translation that causes all polygons to match the other vertex [2].

We use a given k-uniform tiling as the **intersection graph** for a circle packing. That is, the vertices of the tiling are circle centers and the edges of the graph contain the points of tangency for a pair of circles. Because the polygons in a k-uniform tiling all have the same edge length, the circles along the edges of the tiling all have the same radius (namely, $1/2$ the edge length).

Traditionally, Islamic geometric patterns are constructed with a compass and straightedge, with the final pattern dependent on the construction circles and lines. This is evidenced by the construction lines visible in patterns from both the Topkapi Scroll [5] and the Anonymous Persian Compendium [4]. Depending on the number of petals and the complexity of the pattern, sometimes the circles can overlap as in the 10-fold rosette pattern in Figure 2a-c, but in patterns like the ideal 12-fold rosette (2d-e), the circles of the underlying construction are all perfectly situated in a tangent circle packing of the plane.

Our algorithm generates patterns based on **univalent circle packings**, meaning that each circle has a mutually disjoint interior. In order to satisfy the triangulation conditions of [6], we insert an additional vertex and circle in the middle of each non-triangular polygon in the tiling, for example the large circle inside the dodecagon in Figure 3a or the small circle inside the square in Figure 3b.

Most existing star rosette patterns omit the small circle inside the squares of such an underlying tiling, which results in octagons like those in Figure 3a. By including this circle to form a complete circle packing, we allow for 7-pointed stars like those in Figure 3b and other shapes that are uncommon in such patterns.



(a) **(b)**

Figure 3: *Star rosette patterns based on (a) [4.6.12] and (b) [6.4.3.4] uniform tilings. An interactive version of (b) can be accessed at https://www.geogebra.org/classic/ufu67zw9.*

Brewer's Algorithm for Building Star Rosette Patterns

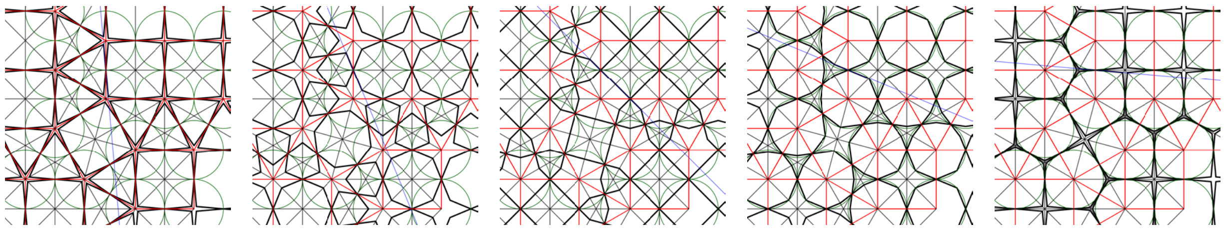
Choose your favorite k-uniform tiling. Construct a circle packing whose intersection graph is that k-uniform tiling, i.e., construct a pattern of circles on that tiling whose centers are the vertices of the tiling and whose radii are half the edge length.

Complete the circle packing in any dodecagon, octagon, or hexagon tile by constructing an interior circle tangent to the ring of bounding circles. Optionally, unless you prefer octagons to four-petaled rosettes, construct an interior circle in any square tile tangent to the four bounding circles.

Draw radii from the center of each circle to each point of tangency with surrounding circles. Construct angle bisectors (red dashed lines in Figure 2) between every pair of radii of tangency. Some of these lines will already exist from the underlying k-uniform tiling.

Construct a line through any point of tangency between two circles that is at a variable angle from the radius of one of the circles. This is the **generating line**, extended in blue through one side of the highlighted petals in Figure 2. The star rosette pattern comes automatically by reflecting this line over alternating angle bisector lines and radii of tangency. Note that it can get quite messy with all of these lines, so we typically only use this method to construct the minimal triangle of a pattern, and then generate the rest of the desired pattern from that minimal triangle. The resulting star rosette pattern fits neatly into the family of patterns described in [3], but allows for arbitrarily fine angle changes, letting us view these patterns as transitions from a k-uniform tiling to its dual, as in Figure 4.

Early versions of this algorithm assumed only star polygons corresponding to the usual division of the circle. For example, if we were constructing a 12-fold star rosette, we would use as our generating line one of the $\{24/k\}$ star polygon lines, with k only taking on integer values from 1 to 11, since the $k = 12$ case is degenerate and 13-23 yield the same stars as 1-11. When we started generating star rosettes in circles bounded by irregular formations, it occurred to us that since it was only the angle of that first generating line that was important, we could throw out the constraint that our star polygon had to be based on a typical number of divisions of the circle. We introduced another slider to vary the n in $\{n/k\}$, as in Figure 3. This was over-complicating things, but we were still stuck in that star polygon mentality. More recent constructions have allowed for a single real-numbered angle that varies from 0 to 90.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 4: *Star rosette patterns based on the* $[(3^3 4^2)^2; 33434; 4^4]$ *uniform tiling.*

Summary and Conclusions

Our method differs from the "polygons in contact" method of pattern construction, championed by [1], in its focus on the circle packing rather than the polygons themselves. Our methodology can be extended to any valid univalent circle packing, though not always with aesthetically-pleasing results. For example, the binary disc packing that inserts a smaller circle in the spaces between any three circles of the hexagonal penny packing [6.6.6] does not allow for star rosettes with equal petals due to the uneven spacing between points of tangency. There is vast potential for circle packing as the basis of pattern generation, especially when combined with other methods of pattern generation including polygons in contact, and allowing for non-uniform tilings of the plane as well as overlapping circle packings.

Acknowledgements

Thanks to Alan Adams for transcribing A.J. Lee's notes, to Daud Sutton for unlocking the inscribed circle key at the first PSTA Geometry Summer School, to Ricardo Hinojosa for so many hours folding IGP's that mirrors are second-nature, to Lena Tran for early contributions to this project, and to our reviewers for helping us shed our over-reliance on the initially-inspiring but ultimately unnecessary $\{n/k\}$ star polygon.

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