Revisiting Ad Quadratum and Ad Triangulum to Generate Hyperbolic Tessellations

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Abstract

Ad Quadratum and Ad Triangulum techniques, developed to provide a proportional order in the medieval period, suggest generative potentials. Reconsideration of Ad Quadratum and Ad Triangulum techniques through the lens of non-Euclidean systems enable the exploration of new pattern generation methods. This study investigates the pattern generation potentials of the Ad Quadratum and Ad Triangulum techniques through hyperbolic tessellations.

Introduction

In medieval architecture, proportions and ratios had crucial importance to provide the divine order [1]. Geometric shapes were arranged in order and symmetry to establish the proportions. The Ad Quadratum and Ad Triangulum techniques are methods based on ratios and proportions between the geometric shapes [4]. These techniques were applied to the plans, sections, and elevations. They were used to determine the width and length ratios of the buildings. Besides, the geometries underlying the structures of the vaults were created with these techniques. Also, patterns developed with these techniques were even used in decoration elements such as rose windows.

The circle is the basis of the geometry used in the Ad Quadratum and Ad Triangulum techniques. Drawing first begins with a circle. By generating multiple circles, geometric shapes are produced from circles' intersections. The relationship between these shapes is copied, and the geometric order is provided. The geometric order underlying these techniques is based on Euclidean geometry. Patterns produced with these techniques can be continuously reproduced and expanded in the desired direction. Also, different patterns with the same base geometry can come together.

Hyperbolic tessellations are patterns that are non-Euclidean and developed within the circle's boundaries like Ad Quadratum and Ad Triangulum techniques. Hyperbolic tessellations have a geometrically specific systematic. However, the geometries included in it are open to improvement. Supporting these forms with different patterns can be used as a methodology in form-finding. This study aims to discuss the productive potentials of Ad Quadratum and Ad Triangulum techniques based on Euclidean geometry when interpreted with a non-Euclidean system. The study construed the generative system provided by medieval techniques through hyperbolic tessellations.

Ad Quadratum and Ad Triangulum

The Ad Quadratum and Ad Triangulum techniques start with a circle. In Ad Quadratum, a square is drawn inside the circle that touches the circle at four points. To create an octagon, the square rotated 45 degrees. Different geometric patterns can be obtained by dividing the squares into smaller squares. The pattern can be copied and reproduced in any direction. The Ad Triangulum likewise begins with a circle. An equilateral triangle is drawn inside the circle that touches the circle at three points. The triangle is rotated 180 degrees around its centroid to create a hexagon. Patterns can be developed by dividing the triangles into smaller triangles.

The proportions developed with Ad Quadratum and Ad Triangulum were used in many areas in Gothic architecture. This study focused on Gothic vault plans developed with these techniques. When the structures in the Gothic cathedrals are examined, the variety of patterns created from a circle stands out. One of the

Gothic vault examples produced with Ad Triangulum is St. Barbara's Church in Kutna Hora. The hexagonal geometry forms the vault structure. The equilateral triangle that touches the circle at three points is rotated 180 degrees around the centroid to obtain the hexagon. Each corner of the hexagon is assumed as the center for new circles. Six circles with equal are drawn, and a floral pattern is obtained. The pattern is copied to match the hexagonal edges (Figure 1).

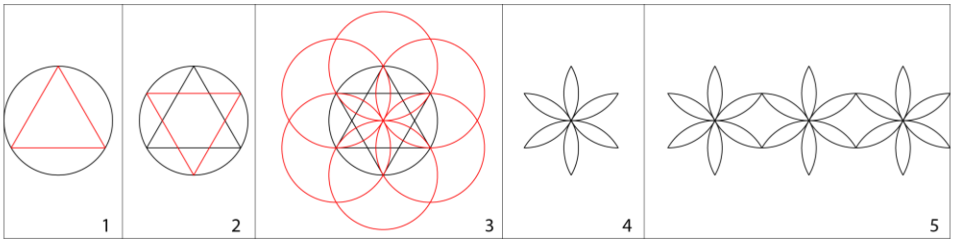

Figure 1: *Ad Triangulum for St. Barbara's Church*

Patterns are developed by various transformations such as division, rotation, and translation. In this way, it is possible to obtain different patterns, although it is produced from a simple geometric shape. In this respect, it can be said that Ad Quadratum and Ad Triangulum techniques are generative techniques for pattern production. Therefore, these techniques have potential in generation and are open to development.

Hyperbolic Tessellations

A tessellation (or tiling) describes the covering of a plane with geometric shapes. The same geometric shape can be brought side by side to repeat continuously, or different geometric shapes can be brought together to produce tessellation. Tessellations are represented by Schläfli symbols. Schläfli symbols are notated as {p,q}. The variable p means that the geometric shape used in the tessellation is a p-gon. The variable q defines how many geometric shapes come next to each other at the vertex point [3]. For example, {3,6} Schläfli symbol of tessellation represents a triangular covering and the combination of 6 triangles at the vertices of the triangles. Similarly, the notation of {6,3} shows the tiling hexagons and the combination of 3 hexagons at the corner points (Figure 2).

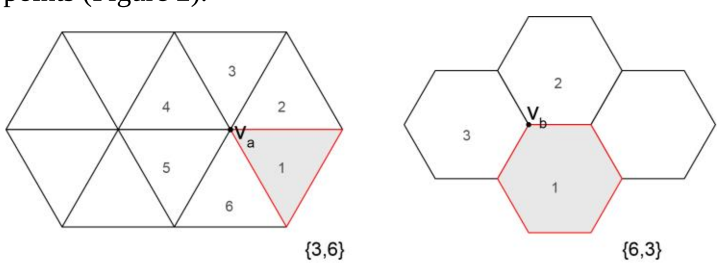

Figure 2: *Tessellations with their Schläfli symbols*

Tessellations can be applied to non-Euclidean planes as well as to Euclidean planes [2]. The tilings applied to elliptic planes, which are non-Euclidean planes, are called elliptical (or spherical) tessellations, and the tilings applied to hyperbolic planes are called hyperbolic tessellations. Although it is not possible to represent the hyperbolic plane, Poincaré defined the disk model to project it in the Euclidean plane. The disk model represents a line as an arc of a circle with ends perpendicular to the disk's boundary.

With a simple mathematical calculation, it is possible to tell whether a tessellation is Euclidean or not. 1/p and 1/q values are summed, and its relation to 1 /2 is examined. According to this mathematical formulation [3]:

If $1/p+1/q=1/2$, {p,q} is an Euclidean tessellation,
If $1/p+1/q<1/2$, {p,q} is a hyperbolic tessellation
If $1/p+1/q>1/2$, {p,q} is an elliptical tessellation

Adhering to this mathematical formulation, let us examine the different combinations of a square. Since the value of {4,3} is greater than 1/2, it is an elliptical tessellation. {4,4} is a Euclidean tessellation because it is equal to 1/2 according to the same formulation, and {4,5} is a hyperbolic tessellation because the result of the formulation is less than 1/2. When the visual equivalents of the tessellations are examined, it is possible to see the validity of the formulation (Figure 3).

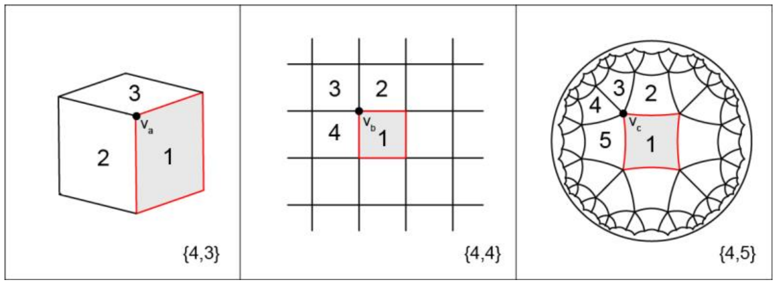

Figure 3: *{4,q} tessellations: {4,3}-elliptical, {4,4}-Euclidean, {4,5}-hyperbolic tessellation*

Pattern Generation

As stated before, Ad Quadratum and Ad Triangulum techniques produce patterns formed by triangles and squares within the boundaries of a circle. The triangular geometric shape is not used alone in Gothic vaults. This shape is used as a basic geometric shape to create hexagons. This study is limited to the regular tessellations of hyperbolic tessellations with {4,q}, {6,q} and {8,q} Schläfli symbols (Figure 4) to produce the counterpart patterns produced by Ad Quadratum and Ad Triangulum techniques in hyperbolic tessellations.

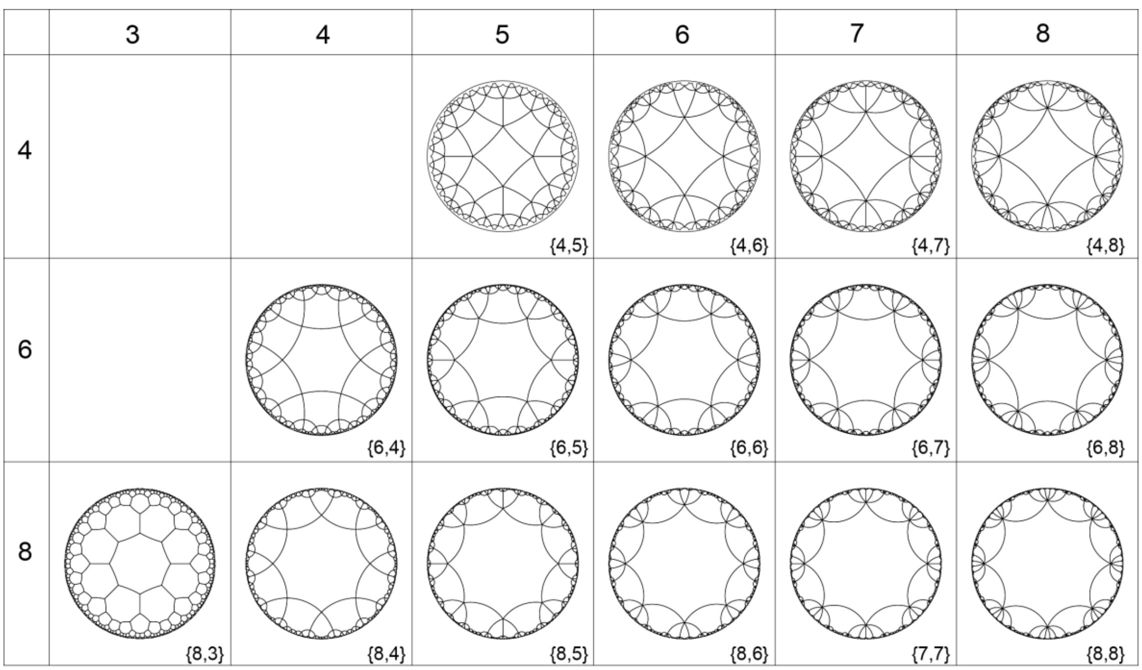

Figure 4: *Regular hyperbolic tessellations with {4,q}, {6,q} and {8,q} Schläfli symbols*

Let us examine Ad Quadratum and Ad Triangulum examples that can be created with hyperbolic tessellations. The St. Barbara's Church vault drawing was created with hexagons. The {6,3} Schläfli symbol can represent it as a Euclidean tessellation. This pattern is obtained when three hexagons are joined by their edges. In this respect, it has the same geometric combination as regular hyperbolic tessellations with {6,4}, {6,5} and {6,6} Schläfli symbols. If the Ad Triangulum technique applied in St. Barbara's

Church is interpreted as hyperbolic tessellation, the tessellations in Figure 5 are obtained. While applying the tessellation, the patterns were drawn on the geometric shapes based on the partitions in figure 4.

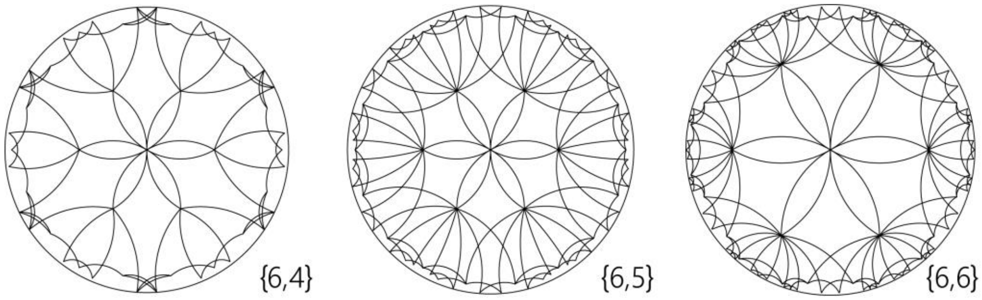

Figure 5: *Hyperbolic tessellations of St. Barbara's Church vault pattern*

The rib vault is one of the most used vault examples in Gothic architecture. Rib vault geometry is based on the square and can be produced with Ad Quadratum. It is an example of Euclidean tessellation and is represented by the Schläfli symbol {4,4}. The four squares are joined by their edges. In this respect, regular hyperbolic tessellations have the same geometric combination as tessellations with {4,5}, {4,6}, {4,7} and {4,8} Schläfli symbols. Hyperbolic tessellation applications of rib vaults produced with Ad Quadratum are shown in Figure 6.

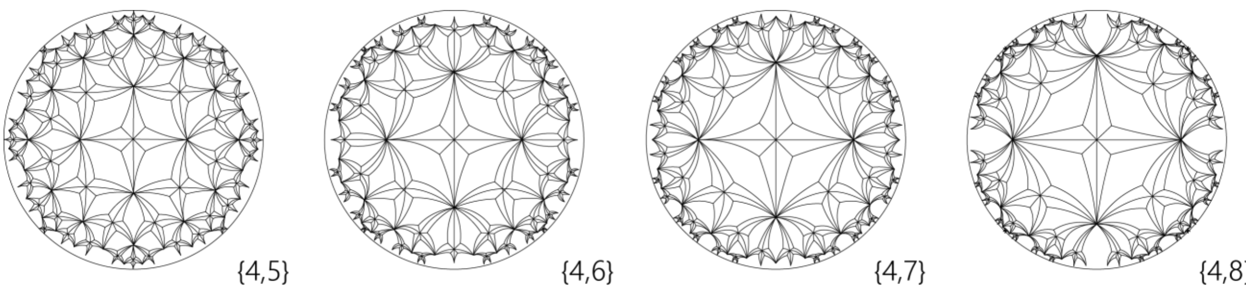

Figure 6: *Hyperbolic tessellations of rib vault pattern*

Conclusion

The Ad Quadratum and Ad Triangulum techniques have productive potential as a pattern production technique and provide a proportional order. The interpretation of these techniques based on Euclidean geometry with a non-Euclidean system enables their productive potential to be considered from another perspective. These techniques discussed in this study are reconsidered with hyperbolic tessellations, a non-Euclidean system. In this context, when the samples examined are reinterpreted on the hyperbolic plane, the tessellations' diversity is striking. We aim to deal with the forms created in the third dimension by the patterns produced for further studies.

References

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