Grafting Tessellations for Fabric Origami

Jiangmei Wu

Eskenazi School of Art, Architecture + Design, Indiana University, Bloomington, USA;
jiawu@indiana.edu

Abstract

This article discusses using a specific tessellation grafting technique to create sewing patterns for fabric origami.

Introduction

In geometry, tessellation refers to covering a flat surface by repeating a geometric shape, or multiple geometric shapes, with no overlaps or gaps. An origami tessellation is often a design in which both the crease pattern and the folded structure use continuous and repeated elements. Origami tessellations are typically folded using paper. The paper is often imprinted with a complex crease pattern made of alternating mountain or valley folds, either by hand or by machine. To fold an origami tessellation, one needs to gently fold and unfold creases in either mountain and valley directions in order to collapse and flatten individual creases—a process that might take hours to complete. There is always a risk of the paper failing or being torn due to this demanding process. Fabric, instead of paper, has been used to fold pleats since ancient Rome and Egypt [3]. Textile pleating is still used today in the fashion industry. Compared to paper fibres, woven cloth fibres are able to move freely or distorted, making cloth an ideal medium to fold intricate, yet easy to produce, tessellations. In fabric origami tessellation, points marked on the cloth are tied and knotted together, allowing the resulting gathering of the fabric to flatten into pleats on the other side. The technique of tying and knotting fabric is also known as smocking, an embroidery technique with its basic honeycomb pattern that was used extensively in the 18th and 19th centuries to provide an elastic quality to non-elastic textile [4]. Today, smocking is used by designers to create specialized decorative patterns that blends traditional craft with modern aesthetics. One of these modern explorations that is rooted in smocking is fabric origami. In fabric origami, points marked on the cloth are tied and knotted together similarly to smocking. However, unlike smocking in which the fabric is left to move freely and wrinkle, fabric origami allows the resulting gathering of the fabric to flatten into pleats on the other side. Origami artist Chris Palmer has used this technique to create a wide array of tessellation design, which he called "Shadowfolds," to be used as decorative textiles [3].

This article explores a tessellation grafting technique to create patterns for fabric origami. It can be said that any origami tessellation can be sewn as fabric origami, however, not all of origami tessellations are suitable for fabric origami. For the sake of the practicality of sewing craft, the study present here is limited to the sewing patterns in which at least three points need to be sewn together to a single point in the fabric.

Grafting Tessellations

To graft a tessellation, one starts by cutting along all edges figuratively and then creating a new tessellation by inserting rectangles, again, figuratively, along all the edges and polygons connecting the vertices. If a vertex in the original tessellation has a valence of three, the polygon connecting the now separated vertices must be a triangle. If a vertex in the original tessellation has a valence of four, the polygon connecting the new vertices must be a quadrangle. In general, for vertices that are n-valent, the inserting polygons must be n-gons. Figure 1 shows a portion of Star Octagram created by Chris Palmer [3]. It is based on an Islamic octangular star that has 8-fold symmetry. Figure 1a shows the original pattern and Figure 1b shows the grafted pattern with the rectangles and new polygons inserted. To make the fabric origami, the corners of each new polygons are sewed together, collapsing the polygons back to points and rectangles to back to lines. On one side of the sewn fabric the seams are now reflecting the original pattern. On the other side of the fabric, fabric is carefully pleated and flattened to create a new tessellation. Figure 1c shows a diagram that is the result of eliminating the polygons of the original tessellation in the grafted tessellation and bringing together the edges of the new polygons, let's call this diagram the grafting tessellation.

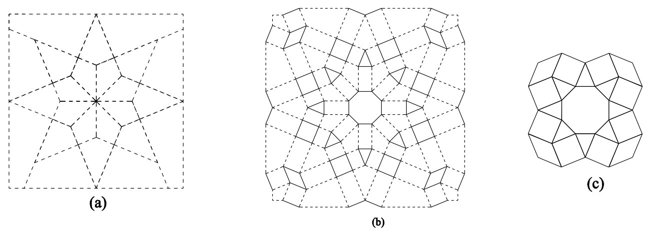

Figure 1: *Grafting of Star Octagram by Chris Palmer inspired by Islamic geometry.*

Primary and Dual Relationship Tessellation Grafting

So one might ask, what kind of tessellations can be grafted? Is there a relationship between the grafting tessellation and the original tessellation? Figure 2 shows another example of grafting a tessellation based on a hexagonal grid. Figure 2a shows the original tessellation (in dashed lines), Figure 2b shows the grafted tessellation, and Figure 2c shows a diagram that combines the original tessellation (in dashed lines) with the grafting tessellation (in solid lines). Figure 2d and 2e show the front and back of the fabric origami that is sewed based on the pattern of grafted tessellation. It can be clearly observed that there is a directly relationship between the original tessellation and the grafting tessellation: the lines in the grafting tessellation are perpendicular to the corresponding lines in the original tessellation. So one might then ask, what is the relationship between these two tessellations? Given any tessellation, is there always a tessellation in which the corresponding lines have a consistent relationship, such as perpendicularity?

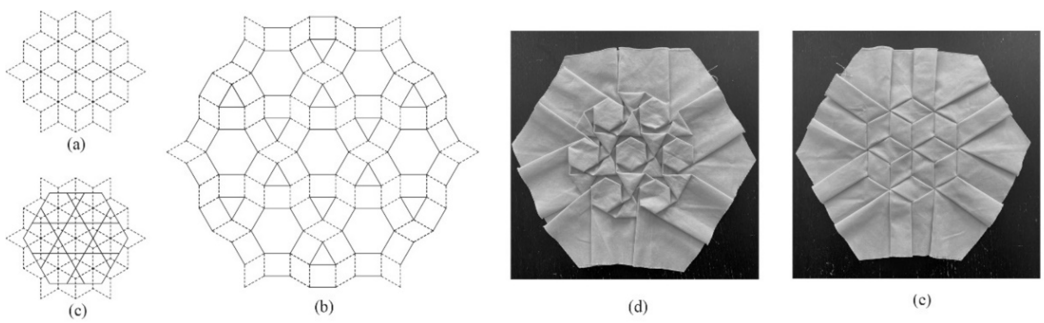

Figure 2: *Grafting of a tessellation based on a hexagonal grid and the resulting fabric origami.*

It turns out that the relationship between these two tessellations has been studied by James Clerk Maxwell (1831-79) [2] and again by the renowned origami artist Robert Lang [1]. Maxwell called the relationship reciprocal as "the properties of the first relative to the second are the same as those of the second relative to the first…" Robert Lang used the understanding of the reciprocal relationship studied by Maxwell to create the primal and dual diagrams for origami tessellations using so-called "shrink-rotate" algorithms [1]. Lang's problem regarding the conditions for "shrink-rotate" algorithms finds an elegant answer in Maxwell's study on how to construct an orthogonal embedding of the interior dual graph for a general plane graph. A given tessellation can be understood as a general plane graph that consists of V vertices, E edges, F faces and Eb of border edges. According to Maxwell, later reiterated by Lang, the grafting tessellation can be understood as its reciprocal figure as an interior dual graph that consists of V' vertices, E′ edges, and F′ faces and has a vertex for every face of primal graph, an edge for every non-border edge of the primal graph, and a face for every interior vertex of the primal graph. The degrees of freedom of the interior dual graph are the coordinates of its vertices, which is $2V'$. The conditions of the interior dual graph are $E'+3$ as each of the edges must be perpendicular to its corresponding edges and there are three additional conditions (two coordinates of a given vertex and the length of one of the edges). According to Lang [1], Maxell's Condition on the possibility of dual graph embedding can be understood as following: if there is exactly one solution for the dual graph embedding, the difference between the degrees of freedom to construct the dual graph and the conditions it requires should be zero. If there are more degrees of freedom than the conditions, then there are more than one solution to construct the interior dual graph. If there are more conditions than degrees of freedom, then it is not possible to construct a dual graph. The difference between the degrees of freedom (*DOF*) of the interior dual graph and its conditions can be expressed in the following expression: $2V' - (E'+3)$ or $2F-(E-E_b+3)$.

Figure 3 shows two different graphs (in dashed lines) and their supposed reciprocal interior dual figures (in solid lines) whose edges are all orthogonal to their corresponding primal graph edges. Figure 3a satisfies Maxwell's Condition with no difference between the DOF and the conditions and therefore has exactly one solution for the interior dual graph. Figure 3b doesn't satisfy Maxwell's Condition and therefore there are no solutions to interior dual graph. However, if we change the original primal graph, as shown in Figure 3c, the interior dual graph can then be closed while satisfying orthogonality. While Maxwell's Condition must be satisfied for there to be a solution for every embedding of a primal graph, there can be a particular embedding if certain symmetric conditions are provided.

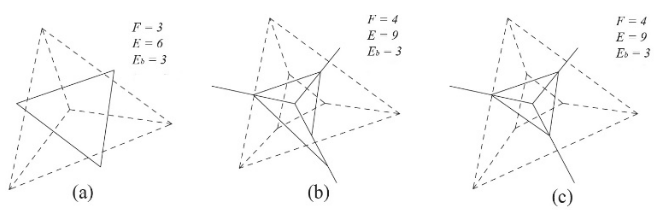

Figure 3: *Three primal graphs and their attempted reciprocal figures.*

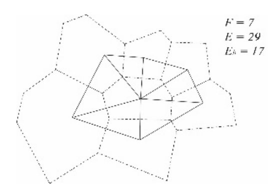

Figure 4: *A Voronoi diagram and its dual.*

Examples of Fabric Origami Using Grafting Tessellations

One class of graphs that doesn't satisfy Maxwell's Condition required to construct reciprocal figures while producing solutions for particular embeddings is Voronoi diagrams and their dual Delaunay triangulations (Figure 4). The edges of the Delaunay triangulation correspond orthogonally one to one with the edges of the Voronoi diagram, therefore, it can be used in grafting tessellation for fabric origami. Figure 5 shows an example of fabric origami based on a Voronoi tessellation. Figure 5a shows the Voronoi tessellation (in dashed lines) with its dual Delaunay triangulation (in solid lines) embedded. Figure 5b shows the grafted tessellation and Figure 5c shows the sewing pattern. Figure 5d and 5e show the front and the back sides of the sewn fabric origami based on the sewing pattern 5c. On the back side of the sewn fabric the seams of the fabric are now reflecting the original Voronoi pattern. On the front side of the fabric, fabric is carefully pleated and flattened to create a new tessellation.

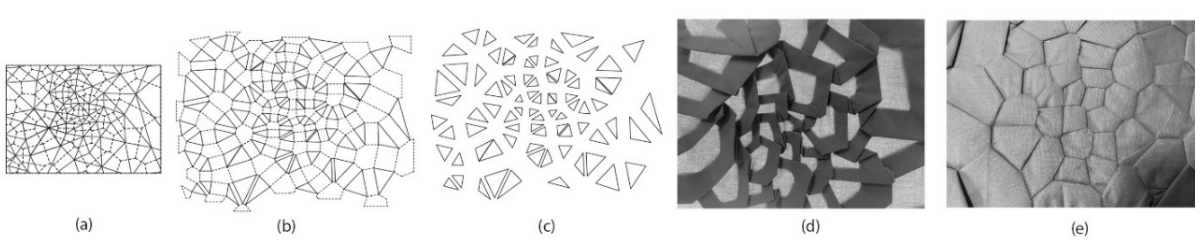

Figure 5: *A Voronoi tessellation, its grafted tessellation, sewing pattern and fabric origami.*

In fact, given any triangular tessellation, one can find the reciprocal figure in which of all of its vertices are 3-valent and the corresponding lines are perpendicular bisectors as the perpendicular bisectors of any triangle meet at one point. Figure 6 shows an example of a triangular tessellation, also called a heptakis heptagonal tiling, based on a mapping of a (7.6.6.) tiling in the hyperbolic plane into the Euclidean plane using *Poincaré* mapping. The reciprocal of the triangular tessellation is a hyperbolic tiling in which there are two hexagons and one heptagon on each vertex. The triangle tessellations are in solid line while the heptagonal/hexagonal tiles are in dashed line. Figure 6a shows the two reciprocal graphs together. Figure 6b shows the grafted tessellation, and Figure 6c shows the sewing pattern. Both Figure 6b and 6c are altered from the original tessellation in Figure 6a by combing the seven triangles in the center into a heptagon for aesthetic reasons. Figure 6d, 6e, and 6f show the front, the back, and the detail of the sewn fabric origami based on patterns in Figure 6c.

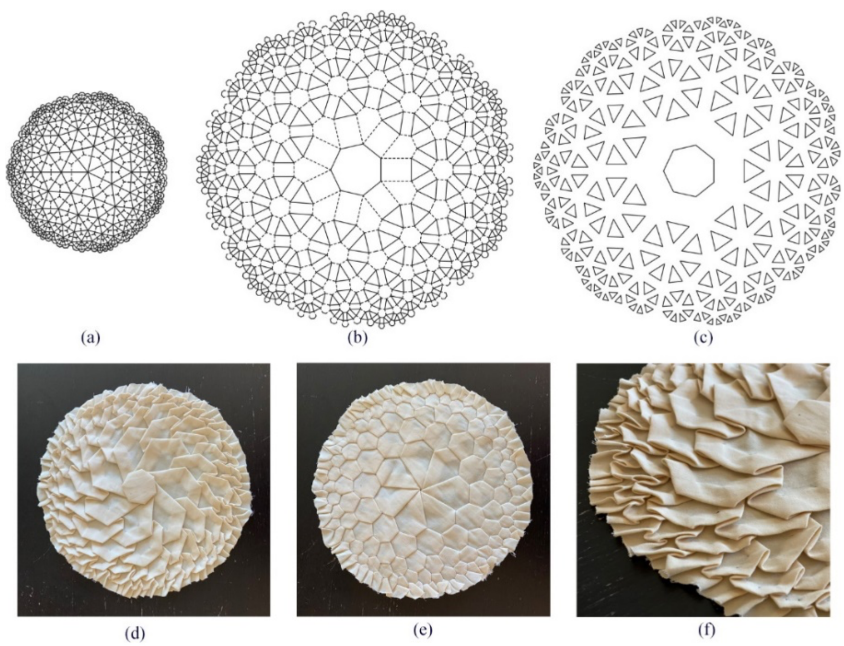

Figure 6: *A triangular tessellation, its grafted tessellation, sewing pattern and fabric origami.*

Conclusions

While there are many more designs in triangular tessellations that can be explored for fabric origami, there are quite a few interesting aspects that need to be further studied. Firstly, this article mostly deals with tessellations with vertices that are mostly 3-valent, thus producing duals as triangular tessellations. How about the tessellations that have vertices that are not 3-valent, but 4,5 or 6-valent? Secondly, there seems to be multiple solutions for the crease patterns. What are the algorithms that determine the flat-foldable crease patterns produced by fabric origami?

References

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