Generalized Pythagorean Lutes

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Abstract

The lute of Pythagoras is a well-known self-similar pattern filling the plane with regular pentagons and pentagrams. This paper describes a generalization to higher dimensions and classification of such objects, yielding lutes corresponding to the dodecahedron in three dimensions and the 120-cell in four dimensions. A semi-formal discussion of these results is presented, together with Zome models and computer images of the new lutes.

The Pythagorean Lute

The Pythagorean lute is a self-similar pattern in the plane formed from regular pentagons and pentagrams arranged around the origin in a decagon and occurring in discrete strata. The ratio of sizes in adjacent strata is the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$. The lute itself comprises ten rotated copies of a single pattern, which we call a *string*, shown in Figure 2 (top). See [2] for a construction.

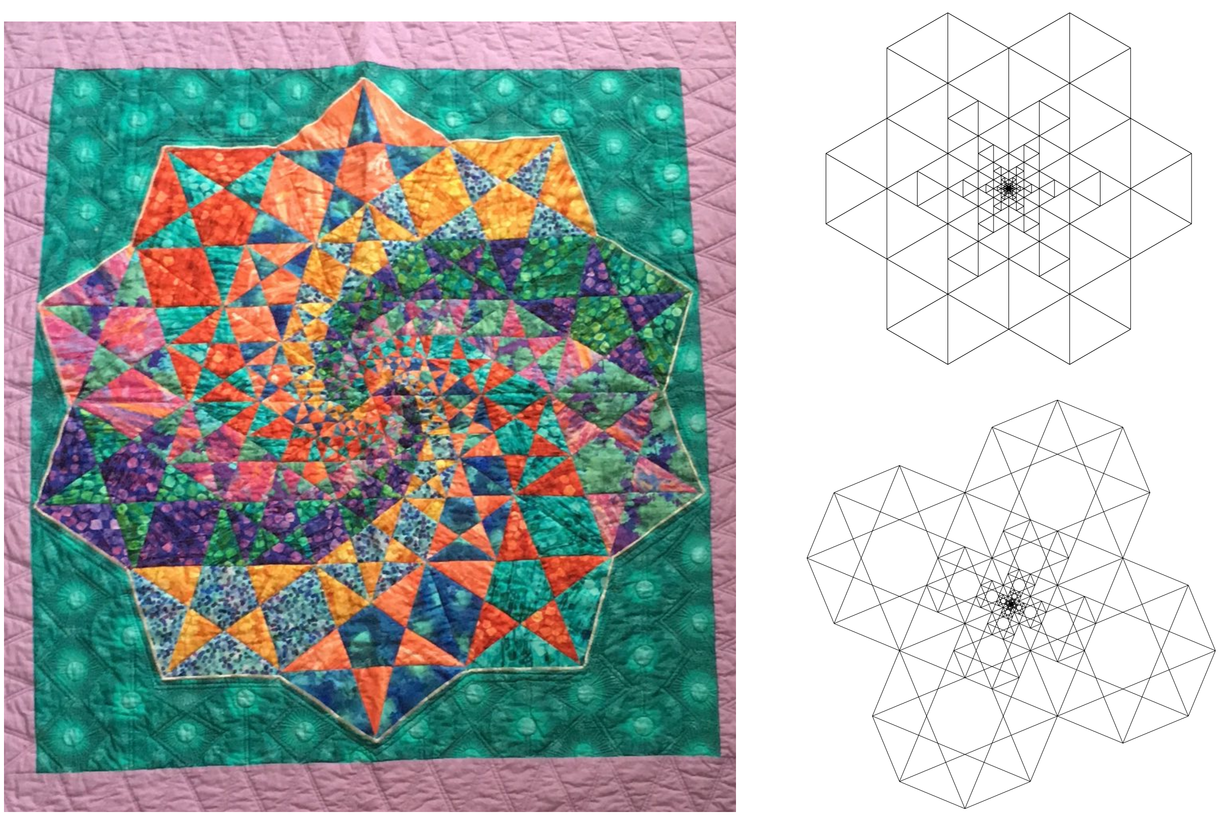

Figure 1: *Left: The lute of Pythagoras on a quilt. Right: the lutes* $(\{6\},\{6\})$ *and* $(\{8\},\{4\})$. *Quilt made by Jacqueline Whitehead (2021).*

Generalized Lutes

The construction of the Pythagorean lute generalizes to n dimensions when polygons are replaced with polyhedra for $n = 3$, and polytopes more generally. The input to the n-dimensional lute construction is two n-dimensional regular polytopes P, Q whose top-dimensional "faces" (for example, edges when $n = 2$ and faces when $n = 3$) are equal. To build a lute, generate a n-dimensional string using P and scaled copies of itself, and then place a single string at each top-dimensional face of Q. Using the Schläfli symbol $\{p\}$ to denote a regular p-gon, the pair $(\{5\}, \{10\})$ is the classical Pythagorean lute.

Not every pair (P, Q) yields a lute: all regular polygons have identical edges, but Figure 2 shows two obstructions to existence of 2-d lutes. One (when P is a triangle or square) is that extending the edges incident to a particular marked one does not result in a cone from which a string can be made. The second is that extending edges in this way may result in a different cone than the one generated by the marked edge and the origin; Figure 2 (bottom right) shows an example where $(P, Q) = (\{5\}, \{4\})$. In fact, to form a lute $(\{p\}, \{q\})$ must satisfy $p \geq 5$ and $q = \frac{2p}{p-4}$. The only integer solutions are $(p, q) \in \{(5, 10), (6, 6), (8, 4)\}$, with size ratios $\frac{1+\sqrt{5}}{2}$, $\frac{1}{2}$ and $\sqrt{2} - 1$. The three two-dimensional lutes are shown in Figure 1.

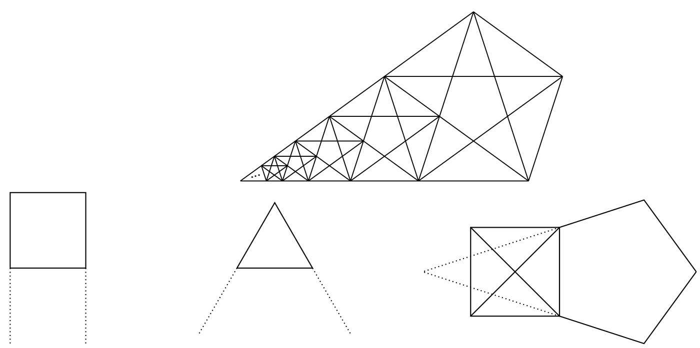

Figure 2: *Top: a single (stellated) string of the* $(\{5\}, \{10\})$ *lute. Bottom: Two classes of obstruction to* 2*-d lutes: triangles and squares do not form strings (left) while strings do not intersect at the origin for most* P, Q *(right).*

The precise notion of an n-dimensional string is somewhat technical. Given a regular polytope P one marks a particular face F and considers the subspace W bounded by the half-spaces containing the faces incident to F. When these half-spaces meet at a single point continued from the line segment based at the centroid of P and passing through centroid of F, then we say an n-string is a collection of scaled copies of P contained in W with centroids on this line segment, that cover the boundary of W and satisfy that no two edges on the boundary have intersection greater than a single point.

A 3-Dimensional Dodecahedral Lute

Let $D = \{5, 3\}$ denote the regular dodecahedron. There exists a 3-d lute (D, D) formed by 12 strings of dodecahedra, with edge lengths in geometric progression of ratio $\frac{1+\sqrt{5}}{2}$. As in the case of the classical lute, this object is displayed most beautifully in composite with a stellation of the polyhedron P; however, the dodecahedron lacks a canonical stellation. One remedy is to ask that as many vertices as possible of the stellated dodecahedron already exist in the complex forming the string. In this sense, the most natural stellation of D is the great stellated dodecahedron, one of the Kepler–Poinsot solids. Of the 32 vertices of this polyhedron, 26 are vertices of the dodecahedra of the string. The stellated dodecahedral string and full lute are shown in Figure 3.

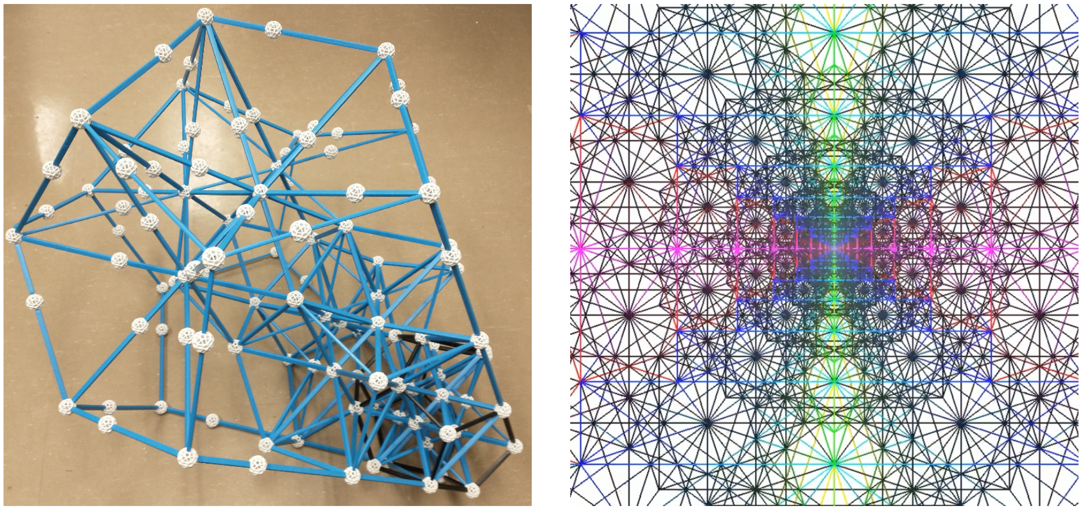

Figure 3: *A partial Zome construction of the stellated* (D, D) *string (left), and the full lute (right).*

Dimension Four and Beyond

The concepts behind the obstructions to existence of 2-d lutes also severely limit the existence of lutes in higher dimensions. No n-d lutes exist for $n \geq 5$; in all cases strings will not exist as we have defined them, for the same reason that strings do not exist for the cube, tetrahedron, and octahedron in three dimensions, and the square and triangle in two. This is not surprising—in both dimensions two and three, the lutes occur with exceptional symmetry, in the sense that the symmetry groups are not those in the infinite families A_n, BC_n, D_n (cf. [3]), and no exceptional symmetry groups in dimensions five and higher yield regular polytopes (see [4]).

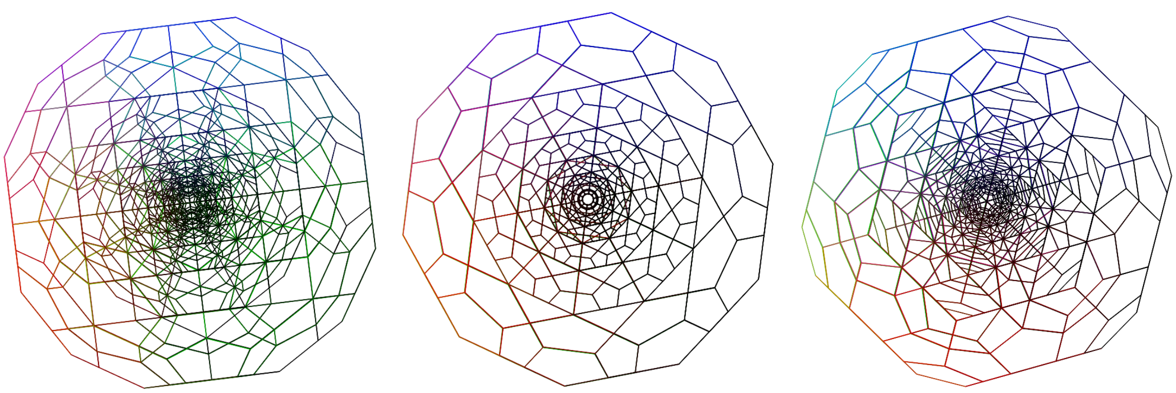

Figure 4: *Three orthogonal projections of the 120-cell string along axes of symmetry.*

What is left then is dimension four, where one finds a 120-cell lute, again with scale parameter $\frac{1+\sqrt{5}}{2}$, the golden ratio. Using the origin-centered 120-cell contained in the Maple package [1], the next 120-cell in the chain is obtained by applying the transformation $x \mapsto \frac{\sqrt{5}-1}{2}x + \sqrt{\frac{3-\sqrt{5}}{4}}e_4$, where e_4 is the fourth standard basis vector. Figures 4 and 5 show a single string of this lute in five projections: three orthogonal ones through the axes of symmetry of the origin-centered 120-cell, as well as stereographic and H_4 Coxeter plane projections.

The actual lute contains 120 of these strings, arranged around the cells of a 120-cell. Animations showing the full lute are available through the companion Git repository, [5]. No other regular 4-polytopes support 4-lutes.

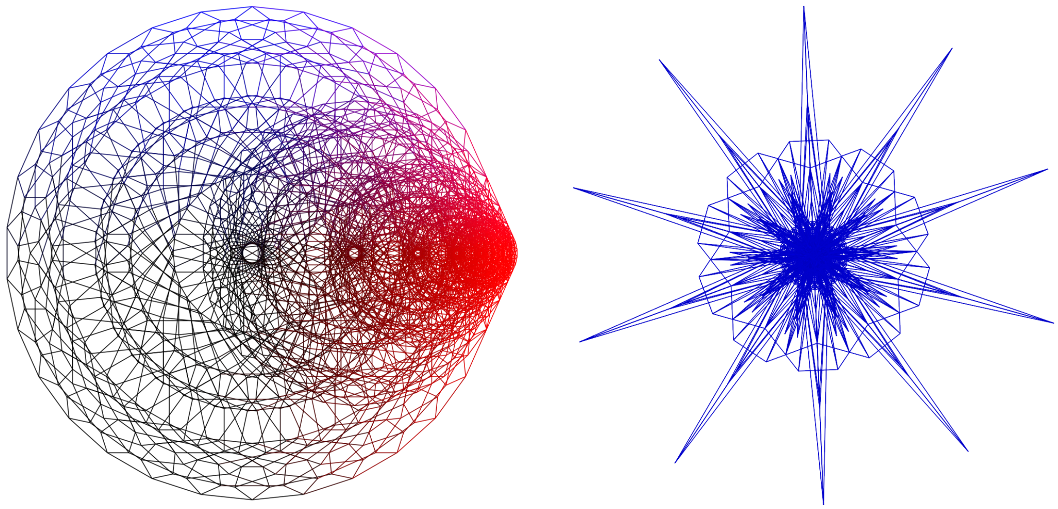

Figure 5: *H_4 Coxeter plane projection (left) and stereographic projection (right) of a single string of the 120-cell lute*

Summary and Conclusions

The pattern of the Pythagoras lute is not merely visually appealing, but one of only three such self-similar 'lute' patterns in the plane. It is a shadow of the exceptional geometry of the group H_4 in four dimensions, and looking at the full H_4 reveals a new 'lute of Pythagoras' in each of dimensions three and four.

Acknowledgements

I thank Benoit Charbonneau for many enlightening discussions, and Jacqueline Whitehead for the wonderful quilt inspiring this submission.

References

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[3] H. S. M. Coxeter. "The Complete Enumeration of Finite Groups of the Form $R_i^2 = (R_i R_j)^{k_{ij}} = 1$." Journal of the London Mathematics Society, vol. 10, no. 1, 1935, pp. 21–25.

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