Photogenic Knot Projections on $n \times n \times n$ Rubik's Cubes

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Abstract

In this paper, I outline one set of criteria for categorizing pixelated projections of knots and use these criteria to define *photogenic* knot projections. I then explain how these photogenic projections can be created on the $n \times n \times n$ Rubik's cube and identify three aspects of photogenic projections of knots – number of colors (c), number of faces used (f), and number of layers on the cube (l) – which can be used to characterize a projection. Finally, I define c,f,l*–photogenic* knot projections and discuss minimizations of some knots over c, f, and l.

Pixelated Knot Projections and *Photogenic* Pixelated Projections

As with many mathematical subjects, knot theory has benefited from changes in representing and encoding information. Specifically, discretization of knots has afforded insight into their nature and characteristics would have otherwise been inaccessible (crossing numbers, determinants, polynomials, etc.) [1]. With this section, I will discuss methods of discretizing knots by pixelating the knot diagram as well as a set of criteria for generating what I call *photogenic* pixelated projections. Pixelating a projection of a knot is relatively straightforward, beginning with a knot diagram, assign color to an array of square cells so that the pixelated image resembles the knot projection. This can be done algorithmically or simply manually and is a convenient way to generate a discretized projection of the knot.

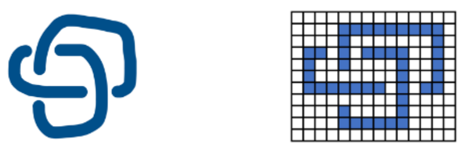


Figure 1: *A typical diagram of the trefoil knot (left) and a pixelated projection of the knot (right)*

This direct pixelation of a knot diagram can use a substantial number of pixels to differentiate the background space between any two segments of the knot, show turns in the projection of the knot, and show the space that indicates over- and under-threads at crossings. Using contrasting colors for the segments can help reduce the number of pixels used by removing the need for spaces at crossings. Further, if one imagines contracting the knot, the empty spaces between threads would eventually disappear entirely. However, this might result in unsightly tangles of color that might result in an indiscernible knot. Because of the possible issues that arise from trying to contract pixelated projections of knots, I propose here a set of criteria that result in what I call *photogenic* pixelated knot projections. As the name implies, these criteria are largely for aesthetic reasons, rather than mathematical reasons, but they do tend to result in knot projections that are discernable and use relatively few pixels.

The following six criteria address issues I have encountered when pixelating knot projections that tend to make the projections more difficult to interpret and the knots more difficult to identify. I call diagrams that satisfy these criteria *photogenic* pixelated knot projections (or simply "photogenic knots"). Figure 2 shows examples and nonexamples to give some intuition regarding the following criteria:

1. Pixels of the knot should contrast with the background and the background color should form a border of at least one row of pixels around the entire knot.
2. Segments of the knot should be composed of straight lines that turn at right angles.
3. Segments should be one color.
4. Segments of the same color should not cross without a gap indicating which segment is over/under.
5. Segments should maintain color "through" the crossing.
6. Parallel threads and crossings near right angles should be spaced apart to show a pixel of background color.

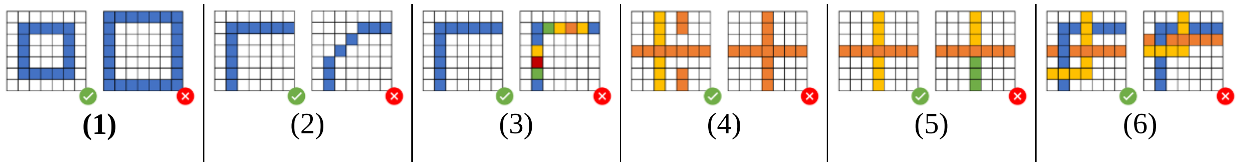


Figure 2: *Illustrating the six criteria for photogenic knots with examples and nonexamples*

Photogenic Projections on the $n×n×n$ Rubik's cube

Although these criteria hopefully provide some general sense of what to expect when encountering a photogenic knot, the reader likely has (reasonable) misgivings regarding the degree to which these criteria clearly define a mathematical category. Admittedly, although I have been generating knots of this type for over a year, I only began trying to define photogenic knots quite recently. Most notably, the reader likely wonders, given criteria (3) and (5), how a knot composed of one continuous thread might have more than one color. This is partially because these criteria are intended for describing pixelated knot projections on an $n×n×n$ Rubik's cube. Specifically, the $n×n×n$ Rubik's cube can serve as a medium composed of 6 adjacent $n×n$ grids of pixels. By permuting pixels between faces of a solved cube one may imagine arranging pixels to represent pixelated knots. If the thread of one such knot crosses from one face to another, then, by the nature of the construction of pieces in the cube, that thread will necessarily change collor. I will discuss this more in this article, but first I will provide some background explaining the structure of the Rubik's cube to help contextualize the composition of knots on the cube.

Since nearly its inception, the Rubik's cube has served as an accessible space for individuals to explore notions of group theory. Cubers (Rubik's cube enthusiasts) use algorithms (sequences of moves) to permute or re-orient specific pieces around the faces of the cube with intended effect. In addition to solving the cube so that each face is a solid color, some cubers have taken to generating patterns on the face of the cube. Indeed, some of the earliest manuscripts on the Rubik's cube described such patterns [2, 3]. Advances in designs and manufacturing have made "larger" $n×n×n$ Rubik's cubes available, ranging in size from 2×2×2 to 21×21×21. Several authors have described various solution strategies for the 4×4×4 and 5×5×5 cubes, which generalize to the $n×n×n$ cubes [4, 5].

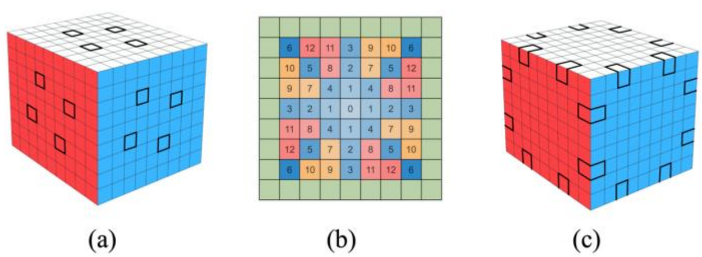


Figure 3: *Permutation classes of center pieces (a, b) and edge pieces (c)*

There are three general types of pieces on $n×n×n$ cubes: corners, edges, and center pieces. These are also the three types of pieces in a standard 3×3×3 cube, except that the $n×n×n$ cube has multiple types of center pieces and edge pieces. Each of the various types of center and edge pieces can be categorized in permutation classes (Figure 3). Figure 3a shows one such permutation class on three visible faces. Any of the highlighted pieces can be permuted with any of the others (as well as 12 such pieces on the reverse of the cube) in 3-cycles. These permutation classes dictate which pieces can be interchanged with each other. For instance, it is impossible for the center-most piece (i.e. 0 in Figure 3b) to be exchanged with a piece nearer the edge of the cube. Figure 3c shows one equivalence class of edge pieces, which can be exchanged with each other. The corner pieces also form their own permutation class and can be re-oriented with a parity restriction on their orientation identical to corners on the 3×3×3 Rubik's cube.

All edge pieces on $n×n×n$ cubes are composed of two colors and are oriented. For instance, on the edge of a 9×9×9 cube where the blue and white face meet, there are 7 edge pieces, a centermost piece and three wing pieces on either side, all of which is unique on the cube. When included in a segment of a pixelated knot, these edge pieces can (indeed must) serve to transition from one color to another, allowing crossings made of two colors as shown in Figure 2.5. Accordingly, photogenic knots on the $n \times n \times n$ Rubik's cube can be composed of multiple colors by allowing the projection to cross an edge. For example, Figure 4 shows a photogenic projection of the Conway knot on a 11×11×11 cube. At each edge color changes, allowing the continuation of the knot onto adjacent faces and allowing the projection to take on different colors on different segments of the projection so that the segments may cross without gaps. Seeing such a projection of the Conway knot on a 11×11×11 cube might lead one to wonder whether this knot could fit onto a smaller cube; whether it might fit on two faces, rather than three; or whether it can be created using only two colors of "thread" rather than three.

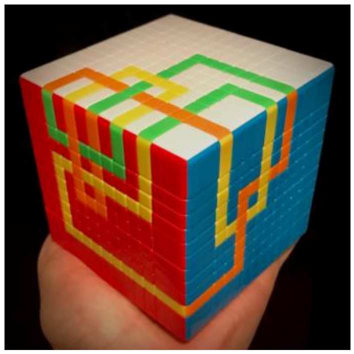


Figure 4: *Photogenic projection of the Conway knot on an 11×11×11 cube*

c,f,l–Photogenic Knot Projections

The above musing about the Conway knot point to three main characteristics of photogenic knot projections on Rubik's cubes: (1) *c* – the number of colors that comprise the thread (not the background), (2) *f* – the number of faces used, and (3) *l* – the number of layers on the cube (colloquially, the "size" of the cube). From this, we can say a knot (or its projection) is *c,f,l–photogenic* if a photogenic pixelated projection of the knot can be composed of *c* colors on *f* faces of a cube with *l* layers. For example, Figure 5 shows three different *c,f,l*–photogenic projections of the trefoil knot: 4,2,9; 6,3,5; and 2,3,9.

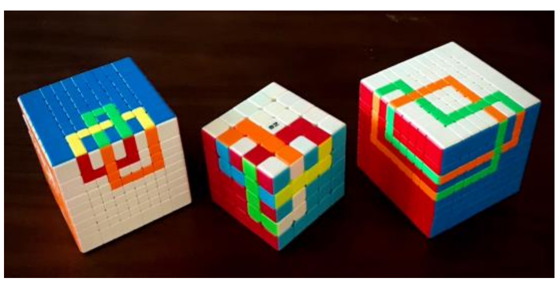
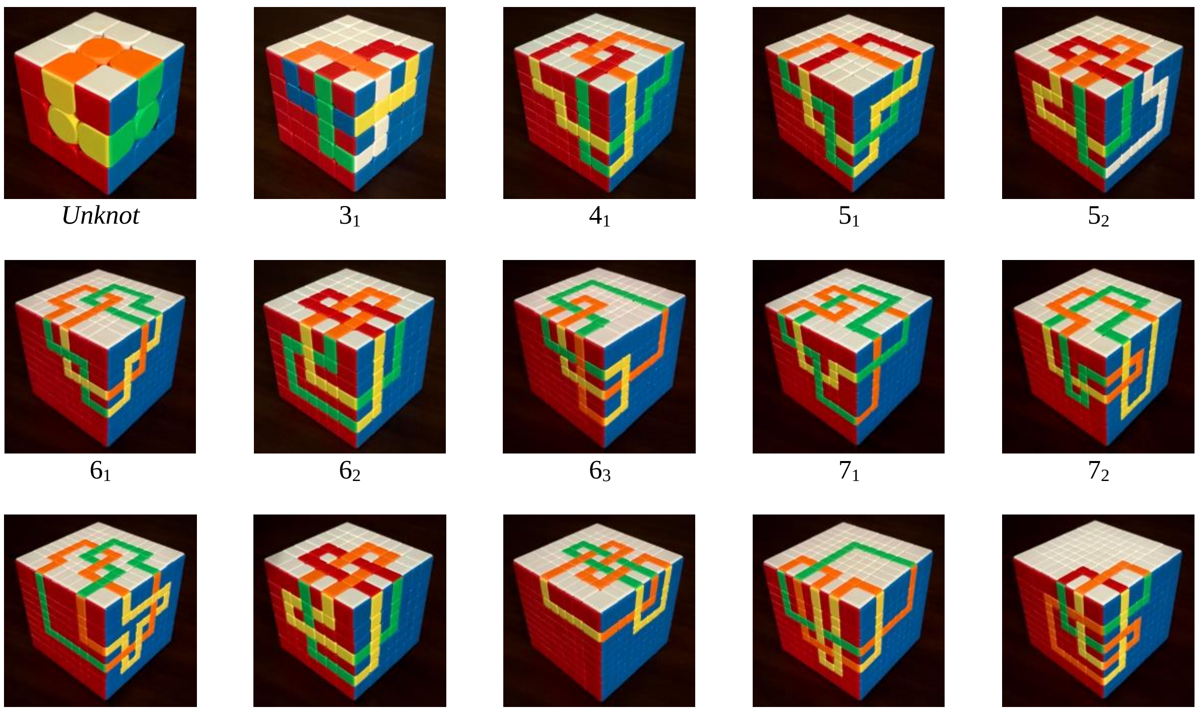


Figure 5: *(left to right) 4,2,9-; 6,3,5-; and 2,3,9-photogenic projections of the trefoil knot.*

The size of a cube and the cube's mechanical construction introduce constraints for whether it is possible for a given knot to be produced on that cube. For instance, the traditional 3×3×3 Rubik's cube does not have enough pixels for an 11-crossing knot such as the Conway knot. However, as shown in Figure 4, it *is* possible to generate a photogenic Conway knot on the 11×11×11 Rubik's cube. Further, as shown in Figure 5, a photogenic Trefoil knot can be generated on a 9×9×9 cube using only two colors. However, producing the Trefoil knot on the 5×5×5 cube *requires* all 6 colors to be used for the thread. The constraints regarding whether a given knot is photogenic on a given cube, using a given number of colors, or a given number of faces are difficult to anticipate. However, in my exploration of these types of knot projections on Rubik's cubes, I have identified several methods for producing projections as well as for reducing the number of colors, faces, or layers required for given knots. Specifically, I have focused on *c*,3,*l*–photogenic projections of all knots up through 7-crossings and have found projections for all of these knots on 9×9×9 cubes or smaller (Figure 6). These photogenic knots are on the smallest sized cubes that I have been able to fit them to date and then minimized for number of colors used. I have focused on *c*,3,*l*–photogenic projections because three faces are the most number of faces of the cube visible from a single perspective. Using three faces also allows one to contrast the background colors of the visible half of the cube with the three colors of the reverse half of the cube. The projections in Figure 6 provide upper bounds for each knot's *c*,3,*l*–photogenic projection.



Unknot, 3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3, 7_1, 7_2, 7_3, 7_4, 7_5, 7_6, 7_7

Figure 6: *c,3,l-photogenic projections of all knots through seven crossings*

In future work, I intend to share more insight into the processes for minimizing *c* and *l* for a given knot's *c*,3,*l*–photogenic projection. This would naturally lead toward trying to generalize, given any knot, what that knot's optimized *c*,3,*l*–photogenic projection(s) might be. For example, it can be shown that the 6,3,5- and 2,3,9-photogenic projections of the trefoil knot in Figure 5 are optimal. That is, the 6,3,5-photogenic projection is the smallest cube on which a trefoil could be generated on three faces and, once embedded on the cube, all 6 colors must be used for the thread of the knot. Similarly, two colors of thread are necessary for any photogenic knot in which threads of the same color do not cross. So, this would be a 2,3,*l*-photogenic projection of the trefoil, which is impossible on any cube with fewer than 9 layers.

References

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