Portals to Non-Euclidean Geometries

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Abstract

We discuss how to create and visualize mathematically accurate immersive portals that link various geometries.

Introduction

Portals are usually openings in walls of buildings, gates, or fortifications that allow entrance to an important structure [3]. While the term originated in the architecture, popular culture and art extended its meaning – portals refer to technological or magical doorways connecting two distant locations. Notable examples include *Through the Looking-Glass* by L. Carroll and the genre *portal fantasy* with works by E. Nesbith or C.S. Lewis [2]. On the one hand, portals solve problems with linking the real world to fantasy worlds. On the other hand, contrary to, e.g., teleportation, they give recipients time to acknowledge the change and accept it. One has to approach the portal and transfer through it to finally arrive at a new place. While teleportation may occur against one's will (recall Dorothy's arrival in F. Baum's *Wonderful Wizard of Oz*), using a portal is consensual; it also enhances the feeling that the two worlds physically border and engage each other.

More recently, portals have gone beyond the children's literature to become a popular trope in video games. While early attempts depicted portals as circular, blurred, flat objects (as in, e.g., Diablo), Portal by Valve introduced *immersive portals* that could be seen through. Similarly, the recent *immersive portals* mod for Minecraft allowed players to place portals at arbitrary locations in the Minecraft world. Sophisticated structures created in this mod started to be called *non-Euclidean Minecraft* even though they do not exhibit the properties of non-Euclidean geometry in the mathematical sense. However, there are games really taking place in non-Euclidean geometries; in HyperRogue [9], the protagonist Rogue, a Euclidean native, explores a world with hyperbolic geometry, which raises the question of how exactly Rogue arrived there. In mathematical visualization, one notable example of a portal is KnotPortal [11], based on the idea of W. Thurston[12], enabling the students to learn about knots and branched covers. In art, one should not forget about immersive portals in M. C. Escher's *Another World*.

In this work, we visualize portals that lead through various Thurston geometries (for an introduction to Thurston geometries, we recommend Bridges papers on immersive non-Euclidean visualization [8, 7, 6, 4, 5, 13]). We discuss which portals are possible and provide instructions on how to create them. Our work draws from both applications of portals: we not only create an immersive experience in VR but (hopefully) increase familiarity with how those geometries work. Starting from the well-known Euclidean geometry, we guide through hybrid and non-hybrid geometries, gradually exploring and taming the strangeness of non-Euclidean worlds step after step.

Which portals are possible?

To help viewers navigate and learn about the geometry, we will build the scene from blocks. In Euclidean geometry, we will use cubes (the cubic honeycomb, Schläfli symbol {4,3,4}). Honeycombs will also come in handy in non-Euclidean geometries; we will build them from cells typical for the given geometry.

We obtain a portal between geometries X and Y when we pick two surfaces P_X and P_Y in X and Y respectively and glue them into a single surface P, so that any object or light crossing P_X appears in P_Y and vice versa. Now, what should we do if we wanted to create a portal from $\mathbb{E}^3$ to $\mathbb{H}^3$? A potential first attempt would be to create a square portal. On the Euclidean slide, we could take the square face of a single cube. But how about the hyperbolic side? One idea would be to use the {4,3,5} hyperbolic honeycomb, which is also built from "cubes with square faces". However, these squares are not Euclidean squares: they have angles $\alpha < 90°$. If we constructed such a portal, any object crossing it would immediately change its geometry (for example, a right angle would become α, and the distances between points would also change). Moreover, the distance between points on the portal surface would differ depending on whether we measured it on the Euclidean or the hyperbolic side. Such a portal would not make physical sense, and it is also unclear how a camera could seamlessly cross it (as the angles change, the camera would also suddenly capture things at different angles). Therefore, we place the following first constraint on our portals: the shape of the portal must be the same on both sides. In particular, both sides must have the same *intrinsic* geometry.

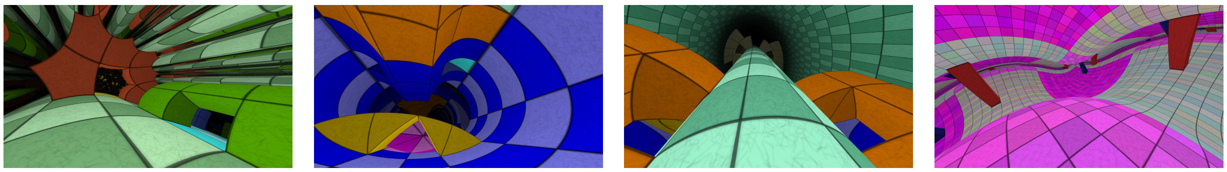
We could solve the intrinsic geometry problem by taking horospheres in $\mathbb{H}^3$. Horospheres have intrinsic Euclidean geometry, so we could cut out a square portal on a horosphere. While this solution solves the crucial problems mentioned in the previous paragraph, we still consider it unsatisfying. To understand why, let us think about real-world mirrors. They resemble immersive portals (unfortunately, only light can cross). The image in a flat mirror accurately shows the geometry of the world on the other side. However, the flatness is crucial – when we consider the curved mirror, the image we see can no longer be considered accurate; it rather corresponds to the curvature of the mirror itself. Since horospheres in $\mathbb{H}^3$ are (extrinsically) curved, the same issue would arise in our horospherical portals. Therefore, we place the following second constraint: the portal must be extrinsically flat (i.e., a totally geodesic surface), or at least, the extrinsic curvatures on both sides must be equal up to the change of sign. Satisfying both constraints at once is impossible for portals between $\mathbb{E}^3$ and $\mathbb{H}^3$. However, the following portals are valid:

Flat portals Totally geodesic surfaces which are Euclidean planes exist in $\mathbb{E}^3$ as well as the product geometries $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. We can construct a honeycomb in product geometries by taking any straight-edged tessellation in its two-dimensional component ($\mathbb{H}^2$ or $\mathbb{S}^2$) and extruding its tiles to three-dimensional cells of height h. For any edge e of such a tessellation, we can make the height h equal to the length of edge e; thus, the extruded edge e becomes a Euclidean square. We can use this square to place a portal to Euclidean space (or the other product geometry).

Hyperbolic portals We can also use portals with intrinsic hyperbolic geometry. One natural choice is to use the right-angled dodecahedral honeycomb {5,3,4} in $\mathbb{H}^3$. This honeycomb has a nice property that its faces are arranged in hyperbolic planes; edges tessellate these planes in the order-4 pentagonal tessellation (just like the faces of the Euclidean cubic honeycombs are arranged into planes tessellated in the Euclidean square tessellation {4,4}). We can use this to create a portal between $\mathbb{H}^3$ with {5,3,4} honeycomb and $\mathbb{H}^2$ with the extruded {5,4} tessellation. Other than regular polyhedra, we can also use the binary tiling of $\mathbb{H}^2$ [1]. The binary tiling has its three-dimensional analogs in $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^3$ and Solv ([10]). Therefore, the tiles of the binary tiling can be used as portals between these geometries.

Spherical portals Portals with intrinsic spherical geometry are based on the same general idea, however, picking a feasible honeycomb is more difficult. The direct analog of the {5,4} hyperbolic tessellation and {5,3,4} hyperbolic tessellation are {3,4} (the octahedral tessellation of the sphere $\mathbb{S}^2$) and {3,3,4} (16-cell in $\mathbb{S}^3$). While the construction is mathematically elegant, the cells of the 16-cell (as well as the right-angled triangular portal itself) are too large in comparison to the size of $\mathbb{S}^3$, so we need other solutions. We can subdivide each of the 8 triangular tiles of {3,4} into six congruent smaller triangles, thus obtaining the hexoctahedron with 48 congruent faces. We can similarly subdivide each cube of the 8-cell into 24 congruent tetrahedra, thus obtaining a honeycomb with 192 congruent cells. Some faces of these cells have the same shape as the faces of the hexoctahedron, yielding a possible portal.

The constructions outlined above let us create portals between six of eight Thurston geometries: $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$, and Solv. For an immersive visualization (especially in VR), it is worthwhile to construct our scene so that the viewer is walking on a floor rather than simply floating through the manifolds. Luckily, it is possible to connect our all six geometries so that they share a common floor, that is, a viewer crossing a portal will still have the same floor (at the right angle to the portal) to walk on.



(a) **(b)** **(c)** **(d)**

Figure 1: *Our portals. (a)* $\mathbb{H}^2\times\mathbb{R}\to\mathbb{H}^3$, $\mathbb{E}^3$*, (b)* $\mathbb{S}^2\times\mathbb{R}\to\mathbb{S}^3$*, (c)* $\mathbb{H}^2\times\mathbb{R}\to\mathbb{S}^2\times\mathbb{R}$, Solv*, (d)* $\mathbb{H}^3\leftrightarrow$ Solv.

Figure 1 (abc) depicts three places in our scene. See https://youtu.be/yqUv2JO2BCs for a live-action video and a link to our interactive visualization. Moreover, interesting visual effects occur when using *recursive portals*, that is, when portals are arranged in such a way that they can be seen through other portals. Figure 1 (d) is an example of such a situation, with square-like fragments of $\mathbb{H}^3$ (pink) and Solv (gray) connected with portals on all sides.

Discussion

A natural question arises whether the two remaining Thurston geometries, i.e., the twisted products of $\mathbb{E}^2$ with $\mathbb{R}$ (aka Nil) and the twisted product of $\mathbb{H}^2$ with $\mathbb{R}$ (aka $\widetilde{SL}(2,\mathbb{R})$), could also be reached. By extruding a straight line in the base two-dimensional geometry of these twisted products, we get surfaces with intrinsic Euclidean geometry. These surfaces have no extrinsic curvature along the fibers; by an appropriate scaling, we could make the extrinsic curvature along the orthogonal direction also match. Thus, we could construct valid portals between Nil, $\widetilde{SL}(2,\mathbb{R})$ and the twisted product of $\mathbb{S}^2$ with $\mathbb{R}$, which is $\mathbb{S}^3$, and thus link all eight Thurston geometries. The problem here is honeycombs: the previously considered regular and Catalan honeycombs in $\mathbb{S}^3$ and the honeycombs obtained by extruding tessellations of $\mathbb{S}^2$ in the twisted product are different. Furthermore, in twisted products, we have less freedom regarding the height of the cells. These issues prevent us from creating elegant portals between these geometries.

Implementation

We render our geometries using a ray-based algorithm and use parallel transport to simulate the camera movements ([10], Sections 3.1 and 4.2). Consider a portal P from geometry X to geometry Y. We introduce the portal coordinates (x, y, z). For points on the portals we have $z = 0$. In the case of Euclidean portals, the coordinates x and y are simply the Euclidean coordinates; for hyperbolic portals we use, e.g., the coordinates in the Klein model. For a point $p \in X$ not on the portals, x and y describe the projection of p on P, while z is the signed distance from P.

A ray r_X in X is rendered using the raytracer in X as usual, until at some time t the ray hits the portal P at point $r_X(t_0) = p$. At each time t, a ray is determined by the point it is passing through $r_X(t)$ and its direction $r'_X(t)$. Since we were rendering X, the coordinates of p are in the natural coordinate system of X. We compute the portal coordinates of $r_X(t_0) = p$, as well as its derivative $r'_X(t_0)$. Then, we compute the natural coordinates of p and the derivative in Y. Thus, we have computed $r_Y(t_0)$ and $r'_Y(t_0)$ in the natural coordinates of Y, and we can continue rendering using the raytracer for Y. A similar approach is used when the camera crosses the portal – this time instead of the derivative $r'_X(t_0)$ we need to compute the tangent vectors corresponding to the "forward" and "upward" directions.

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