Spiral Ruled Surfaces

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Abstract

The hyperboloid of one sheet, being so easy to construct by setting slanted supports at a uniform angle from a circle, is ubiquitous in mathematical art. Can the circle be replaced by a spiral? We show affirmative answers to this question and propose a generalization of the hyperboloid that encompasses quite a variety of beautiful ruled surfaces.

At a meeting of BAAM! (Bay Area Art and Math!), Roman Kogan showed a delightful 3D print of a base in which two rings of pencils may be inserted to create two hyperboloids of one sheet, ruled in opposite ways. The effect is similar to Figure 1, which was created in Rhino with Grasshopper. John Edmark asked whether the same construction could begin with a spiral rather than a circle. My work on the answer led to a formula that produces quite a variety of generalizations, including all three known doubly-ruled surfaces.

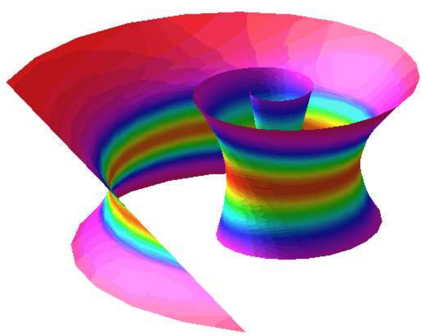
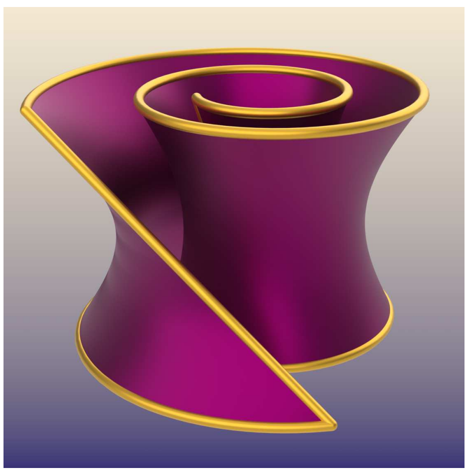

Figure 1: *Two examples of a ruled surface based on a logarithmic spiral.*

From hyperboloid to logarithmic spiral surface

A ruling parametrization of the hyperboloid of one sheet is known to be

$$\vec{x}(u,v) = (a\cos(u), a\sin(u), 0) + v(-a\sin(u), a\cos(u), c).$$

For each fixed value of u, the formula parametrizes a line through a point on the circle of radius a centered at the origin, inclined at a slope c/a. Note that the horizontal component of the vector that generates the line is along the tangent vector to the circle.

Using (x, y, z) as the components of a point on the surface, some algebra shows that every point satisfies

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1,$$

the well-known equation satisfied by points on the hyperboloid of one sheet. (This is a family of surfaces with two parameters, a and c, which govern whether the hyperboloid is tall and narrow or short and wide.)

To replace the circle with a logarithmic spiral, we use the formula for the spiral and its velocity vector

$$\vec{\alpha}(t) = \left[e^{kt}\cos(t), e^{kt}\sin(t), 0\right] \quad \vec{\alpha}'(t) = \left[e^{kt}(-\sin(t) + k\cos(t)), e^{kt}(\cos(t) + k\sin(t)), 0\right].$$

Creating rulings just as we did for the hyperboloid and condensing with algebra leads to a parametrization

$$\vec{x}(u,v) = \left[e^{ku}(\cos(u) + v(-\sin(u) + k\cos(u))), e^{ku}(\sin(u) + v(\cos(u) + k\sin(u))), cv\right],$$

where c is a parameter that controls the initial ($u = 0$) steepness of the rulings. (Note that if $k > 0$ the rulings approach vertical lines for large negative u; they approach horizontal lines for large positive u.)

Not having mastered all the possibilities of rendering in *Rhino*, I imported the OBJ files into *Photoshop* and decorated them there. On the left in Figure 2 is a spiral ruled surface textured with a rich wood grain and brass edges. Bringing these virtual objects into the real world required thickening the surfaces. *Rhino* made that easy and my friend Mario Micheli produced the elegant 3D print on the right in Figure 3.

Figure 2: *Left: Wood grain on a spiral ruled surface with brass railings. Right: A 3D print by Mario Micheli, who also supplied the photograph, with a key to show the scale.*

When John Edmark saw my images, he asked whether I could cut the rulings so that all segments had the same length. His question inspired me to create the designs in Figure 3. For these surfaces, depicting only the top half ($v \geq 0$) allows the virtual object to sit on a flat base. On the right in Figure 3, I used a shortened list of u-values and created pipes along the rulings. Both the smooth surface and the piped version show promise as sculptures.

Generalizing to other curves

To create the spiral surface, I had observed that the rulings of the hyperboloid were lines along a vector with one component along the tangent of the generating circle and another in a transverse direction. If we start with a planar generating curve $\vec{\alpha}(t) = (x(t), y(t), 0)$, with velocity vector $\vec{\alpha}'(t)$, the natural generalization is a surface parametrized by

$$\vec{x}(u,v) = \vec{\alpha}(u) + c_1 v\vec{\alpha}'(u) + (0,0,c_2 v).$$

The parameters c_1 and c_2 can be adjusted for aesthetic purposes. They affect the slope of the rulings, but so does the velocity of the curve. In the example of the logarithmic spiral, the speed is proportional to the magnitude of the curve, which leads to a graceful change in the slopes of the lines.

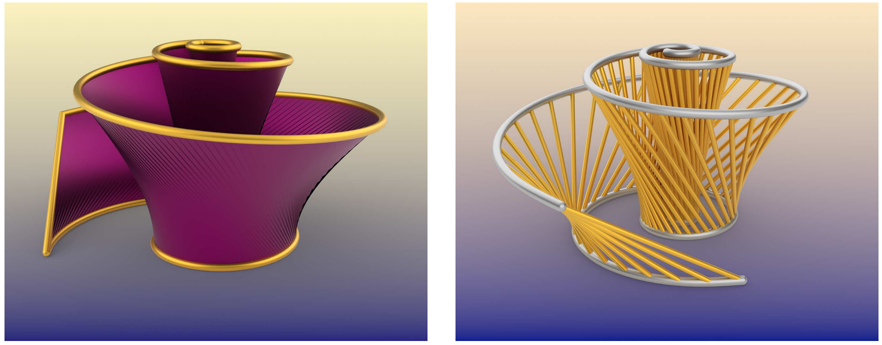

Figure 3: *The top portion of a ruled spiral surface with rulings of uniform length. Left: As a surface. Right: With pipes long the rulings.*

When I saw my first examples of these lovely shapes, I imagined that it would be easy to create a double spiral by pasting together two pieces of the same single spiral surface, suitably rotated. I leave it to readers to investigate this; I convinced myself that it is a foolish pursuit. Instead, I applied the general recipe to a very nice double spiral obtained using stereographic projection and a Möbius transformation. I first transferred the logarithmic spiral to the sphere by stereographic projection, where it spirals symmetrically from the south pole to the north pole. Then I tilted the sphere by a quarter turn and used inverse stereographic projection.

The resulting curve is most easily explained in complex notation. The logarithmic spiral can be written as

$$z(t) = e^{kt}\left(\cos(t) + i\sin(t)\right) = e^{(k+i)t},$$

where $i^2 = -1$. The real and imaginary parts of a complex number are represented as the horizontal and vertical displacements in the plane, so this is nothing more than a repackaging of the familiar Cartesian coordinates. After the transformation described, the new curve is

$$\hat{z}(t) = \frac{1 - e^{(k+i)t}}{1 + e^{(k+i)t}}.$$

It's quite marvelous how the complex framework allows us to keep track of some very complicated formulas rather simply. Moreover, *Grasshopper* is built to accept complex numbers. Figure 4 shows two examples of this lovely shape. In the *Photoshop* version, only the top half ($v \geq 0$) is shown; the texture map uses a photograph of an olive-wood coaster.

Figure 4: *A double-spiral ruled surface, rendered in Rhino (left). Rendered in Photoshop (right).*

The possibilities are as rich as one's imagination in thinking of curves to play the role of the base curve $\vec{\alpha}(t)$. For this investigation, I returned to the symmetric curves I wrote about in 1996 [2]. Figure 5 shows just two possibilities, each a variation on

$$e^{it} + ce^{int},$$

where $n = 4$ on the left and $n = -2$ on the right. Since both coefficients are congruent to 1 modulo 3, the curve has 3-fold symmetry.

It might be tempting to imagine that the surface on the left in Figure 5 might be doubly ruled. However, it is known that there are only three classes of doubly ruled surfaces: the plane, the hyperboloid of one sheet, and the saddle surface. We invite the reader to construct the saddle surface, $z = x^2 - y^2$, using the method described in this paper. It works out nicely.

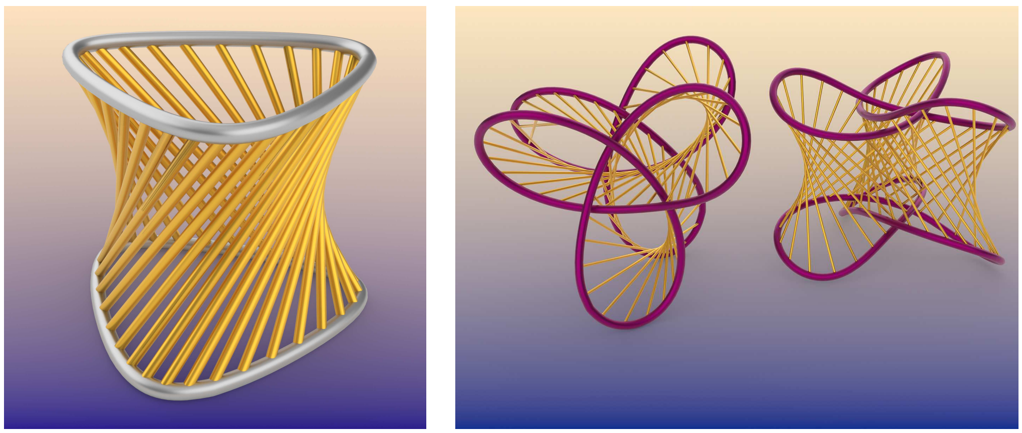

Figure 5: *Left: A ruled surface from a rather plain curve with planar top and bottom. Right: A ruled surface on a self-intersecting curve with bars of uniform length.*

Acknowledgements

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References

[1] F. Farris, *Creating Symmetry, the Artful Mathematics of Wallpaper Patterns*, Princeton University Press, 2015.

[2] F. Farris, "Wheels on Wheels on Wheels—Surprising Symmetry." *Math. Mag.*, vol. 69, no. 33, 1996, pp. 185–189.