Closed Surface Envelopes

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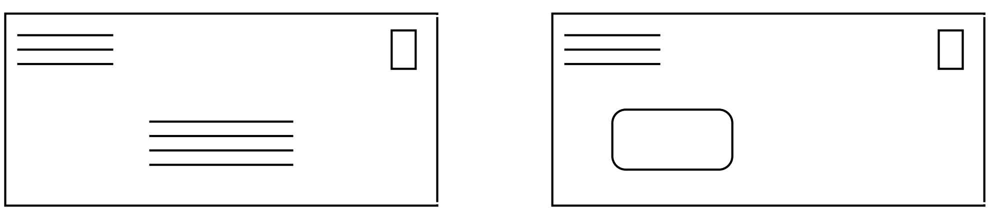
Abstract

I describe some of my mail-art envelope constructions made with paper and glue. Although they are flat (and can be sent through the mail with proper postage), they are nonetheless topologically accurate realizations of compact surfaces such as tori, projective planes and Klein bottles.

This note describes some paper envelopes that I have made for the amusement of myself and my colleagues and friends. Each begins with a LaTeXsource file, which is then typeset, printed, cut, folded and glued. The result is a flat—but topologically accurate—realization of a closed surface (torus, projective plane, Klein bottle, etc.) that can be sent through the mail.

Recall that a *closed surface* is a finite surface that has no boundary. The classification theorem for closed surfaces states that any orientable closed surface is an n-holed torus (that is, a sphere with $n \geq 0$ handles). Further, any *non-orientable* closed surface is a sphere with $n \geq 1$ cross-caps. (A *cross-cap* is a hole capped with a Möbius band.) As an aid in identifying a closed surface S, we can compute its *Euler characteristic*, which is an integer $\chi(S)$ found as follows: Divide S into polygonal regions, so that there are r regions, e edges and v vertices. Then $\chi(S) = v - e + r$. If S is an n-holed torus, then $\chi(S) = 2 - 2n$; If S is the sphere with n cross-caps, then $\chi(S) = 2 - n$. For details, see any text covering surface topology, such as [1].

From a topological point of view, a standard (sealed) mailing envelope—like the one in Figure 1(a)—is a sphere, in the sense that its paper surface can be continuously deformed into a spherical surface. (Envision inflating the envelope like a balloon.) Other envelopes—like the one in Figure 1(b)—have windows. Doubtless you received one in the mail during the past day or so. Such an envelope is a sphere with a hole cut out of it.



(a) (b)

Figure 1: *Standard envelopes. A sphere* (a) *and sphere with a hole* (b).

The window is a great opportunity to turn the envelope into a more exotic surface. Glue a folded strip to it, as shown in Figure 2. An ant crawling on the exterior of the envelope could move onto the strip, across and under the diagonal fold, and arrive on the interior of the envelope. So adding the strip has resulted in a one-sided surface. This is a sphere with one cross-cap, also known as the projective plane. Granted, it has a hole in it (outlined with a thick black line in Figure 2) but this is necessary because the projective plane cannot be placed in 3D space without intersecting itself (or having a hole allowing it to pass through itself).

Adding *two* strips can result in a Klein bottle (with a hole), as described in Figure 3. Figure 4 shows a perhaps more familiar realization of a Klein bottle.

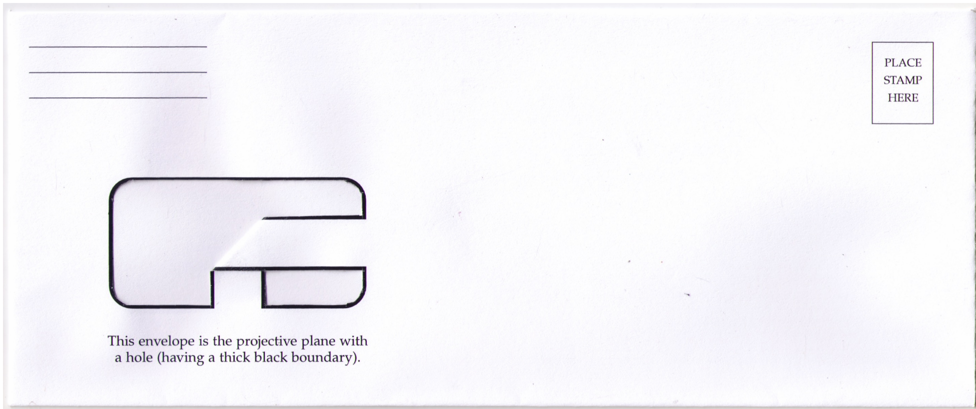


Figure 2: *Adding a folded strip to the window changes the envelope from a sphere (with a hole) to a projective plane (with a hole). Technically it is a Möbius band with a portion thickened to form the body of the envelope. The Möbius band's boundary is outlined with the thick black contour. Capping this with a disk would yield a sphere with a cross-cap, that is, the projective plane.*

Figure 3: *A sphere with two cross-caps, also known as a Klein bottle. It has one hole, outlined in black.*

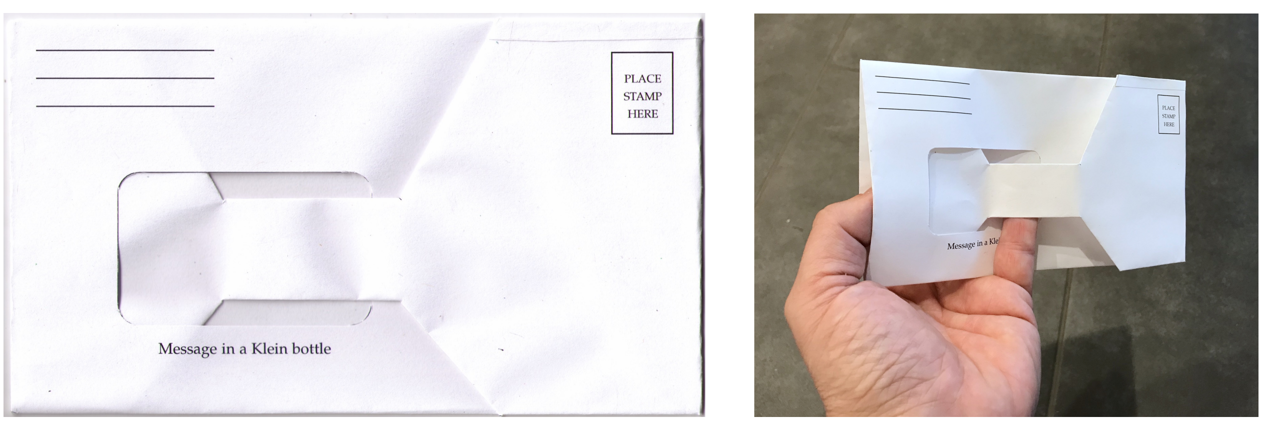


Figure 4: *A flat Klein bottle. Printed on a US legal-size sheet of paper, it is cut and folded lengthwise into a flat tube with a window on its left side and notches on its right side. It is then folded again along a central vertical axis, so that the notched portion can pass through the window. The right end of the tube then meets the left end. Flaps with glue at tube's left end attach to the right end.*

In theory we could paste arbitrarily many folded strips to the window. As explained in the caption to Figure 5, appropriate placement of n strips results in a sphere with n cross-caps (and a hole). By the classification theorem for closed surfaces, we can thus make a windowed envelope that is any non-orientable closed surface we desire.

However, the placement of the strips is significant. Figure 6 shows an envelope with a hexagonal window with three interlocking strips forming three holes. As explained in the caption, the resulting surface is the the sphere with one cross-cap, that is, the projective plane. Indeed, this is a flat variant of Boy's surface, an embedding of the projective plane in 3-dimensional space without pinchpoints.

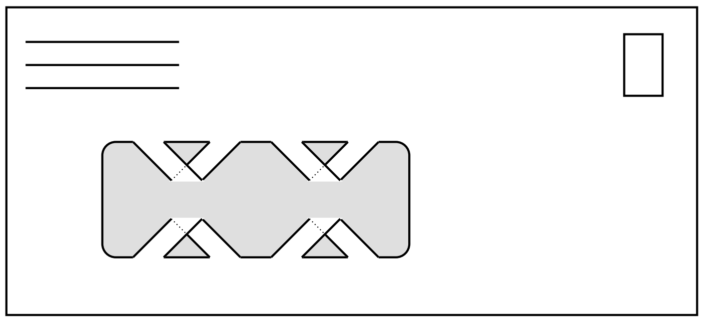


Figure 5: *Add n strips around the window in the manner described here (for $n = 4$). There is precisely one hole (with a black line boundary). The resulting surface is one-sided. What surface is it? To see, compute its Euler characteristic. View the n strips as (folded) rectangles whose four vertices meet the window boundary. Then the hole's boundary is a $4n$-cycle, and the body of the envelope (minus the strips) is a $4n$-gon. Cap the hole with another $4n$-gon. There are now $4n$ vertices, $6n$ edges and $n + 2$ faces. The Euler characteristic is then $\chi(S) = 4n - 6n + (n+2) = 2 - n$. By the classification theorem for closed surfaces, the envelope is a sphere with n cross-caps.*

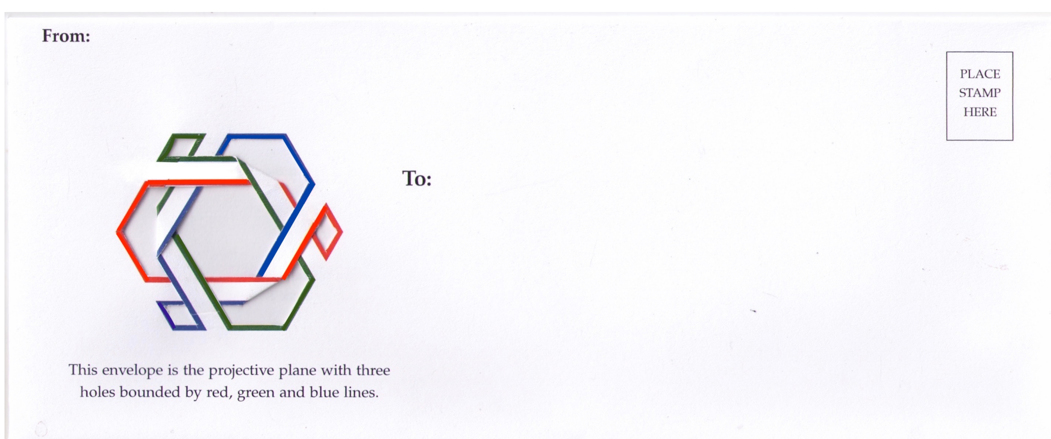


Figure 6: *This envelope E is the projective plane with three holes, bounded by red, green and blue lines, respectively. It is based on Boy's surface, an immersion of the projective plane into $\mathbb{R}^3$ without pinchpoints. Indeed, widen the three strips, and the three holes will narrow to three slits through which the surface passes through itself without singularity. To verify that the envelope is really the projective plane, compute its Euler characteristic. View the three strips as rectangles and cap the three holes with hexagons. The body of the envelope (minus the strips) is then an 18-gon. There are 18 vertices, 24 edges and 7 faces, so the Euler characteristic is $\chi(E) = 18-24+7 = 1$. The projective plane is the only surface with Euler characteristic 1.*

We have so far concentrated on the non-orientable surfaces. I also have a toroidal (orientable) envelope, described in Figures 7 and 8. It opens along a flap on the top, allowing for the insertion of a cylindrical letter.

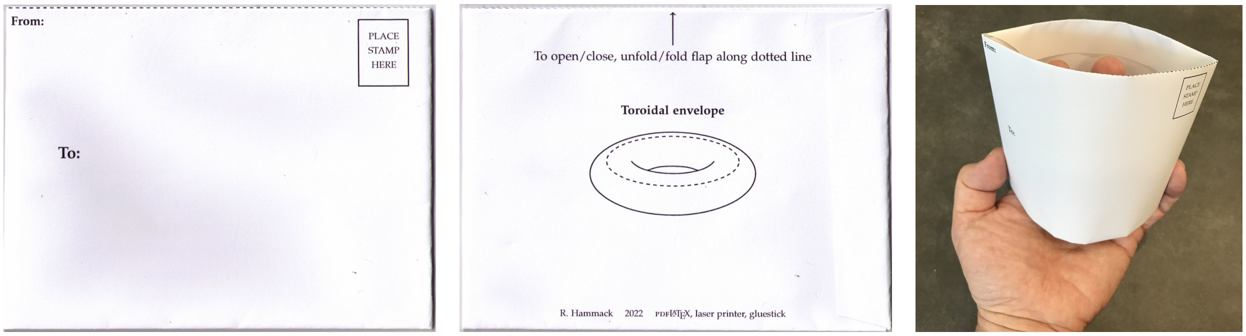


Figure 7: *Toroidal envelope, front, back and selfie.*

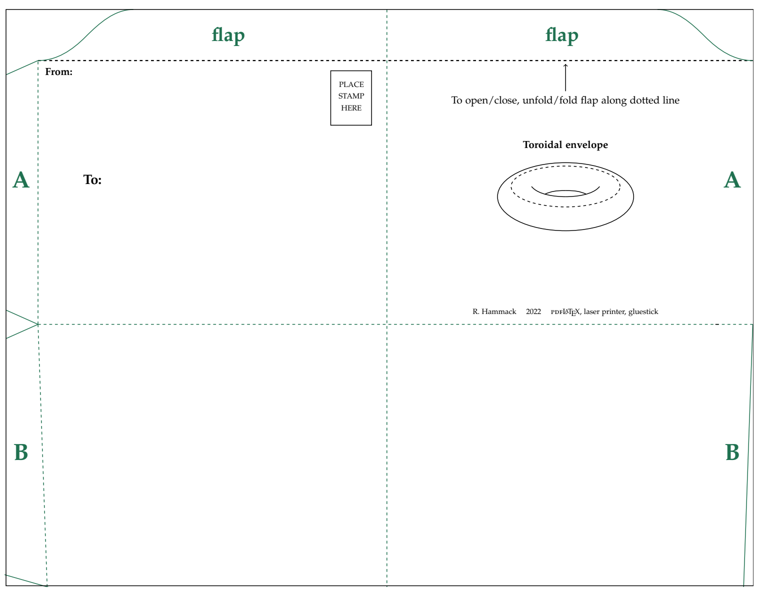


Figure 8: *The toroidal envelope in Figure 7 is printed on an 8.5"×11" sheet, and cut on the solid green lines. It is then folded along the central horizontal dashed line, with text facing out. Next it is folded along the central vertical dashed line (with text facing out). Flap A is glued to side A on the right; Flap B is glued to side B. The torus is sealed by folding the upper flap to the inside.*

Friends and colleagues say getting a closed surface envelope in the mail made them laugh or smile. I take this as evidence that my envelopes have had a measurably more positive effect on the world than have some of my other research projects.

Acknowledgements

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References

[1] S. E. Goodman. *Beginning Topology*, Thompson Brooks/Cole, 2005.