Triply Periodic Links

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Abstract

There have been previous accounts of geometric configurations of linked rings that are periodic in three dimensions. In particular three examples of space-filling toroids derived from regular lattices are known. A fourth is described here. In two of the three known examples the toroids can be replaced by circular tori, which is also possible with the fourth. A completely different kind of triply periodic link can be created using an approach based on Holden's polylinks, which are finite polyhedral arrangements of linked polygons. The extension to infinite polyhedra has not been considered before.

Introduction

In mathematics the word "link" has a particular meaning so I will use *ring* to mean what is normally called a link, and *link* for a collection of rings that cannot be separated without cutting any of them. This is a topological definition so the rings could be any shape. In particular they need not be circular. Usually only finite links are considered but an infinite number of rings is possible. The simplest example is a chain that is infinite in one dimension, but the link could be infinite in two dimensions and extend over a surface, like chain-mail, or in three dimensions and extend throughout space. Only those links that are periodic in three dimensions with a single kind of regular ring will be considered here. More precisely they have cubic symmetry and all the rings are regular and equivalent under the symmetries of the link.

Space-filling Toroids

Carlo Séquin has described space-filling polyhedra of genus 1, toroids**,** which he derived from regular lattices following a method communicated to him by John Conway [10]. A *lattice* is a symmetrical arrangement of *nodes* joined by *edges* in three dimensions. A *circuit* consists of a series of consecutive edges that begins and ends at the same node. A *minimal circuit* of a lattice is a circuit having the smallest number of edges. If minimal circuits are considered to be faces, possibly but not necessarily planar, then a lattice can be seen as a space-filling of polyhedra (which will be saddle polyhedra if their faces are non-planar). A *dual lattice* is constructed by associating a dual node with each polyhedron. Since we are concerned only with symmetrical examples it can be defined as the centroid of the polyhedron. Dual edges connect the dual nodes whenever the associated polyhedra share a face.

The more well-known *polyhedral dual* associates a dual vertex with a polyhedral face. Dual vertices are joined by dual edges whenever associated faces of the polyhedron share an edge. In the symmetrical infinite polyhedra to be considered later all the faces are planar, and the dual vertices can be defined as the centroids of the faces, preserving the symmetry. Actually in these regular cases the dual vertices can be moved continuously, rotating the dual faces about their diagonals, so they become non-regular [3].

Conway's construction begins with a lattice edge and the nearest dual edge. Complete a tetrahedron by joining the nodes at the ends of the edges. The tetrahedron is then divided into two, in the simplest case by the plane that goes through the mid-points of the four constructed lines, so it is parallel to the two lattice edges. Merging all the half-tetrahedra around a single minimal circuit creates a toroid. The toroids will be congruent when the lattice is self-dual. There are four self-dual lattices with cubic symmetry [14].

Cubic

The most obvious regular lattice corresponds to the edges of a space-filling of cubes. It has six edges at each node in three mutually perpendicular directions. The faces created by planar division of Conway's tetrahedra are those of Coxeter's regular sponge with six squares at a vertex [8], an example of a skew apeirohedron, also known as an infinite polyhedron [14]. In all the examples to be considered the vertices are congruent, so the infinite polyhedra can be identified by their vertex symbols, in this case (4^6).

In fact any division of the tetrahedron that produces two identical pieces will ensure that the toroids are congruent, and the division creates patches that combine to form a surface that bisects space into congruent halves. The Schwarz P surface is the one having the minimum area, and Séquin describes finding a toroid in 1995 using approximations to Voronoi zones. Two more examples can be derived from space-fillings of Archimedean polyhedra. The most obvious is Coxeter's polyhedral dual, created from a space-filling of truncated octahedra by omitting the square faces, four hexagons at a vertex, so (6^4). The other is the infinite polyhedron created by omitting the octagonal faces from the space-filling of truncated cuboctahedra and octagonal prisms (Figure 1), three squares and a hexagon at a vertex, so (4^3,6).

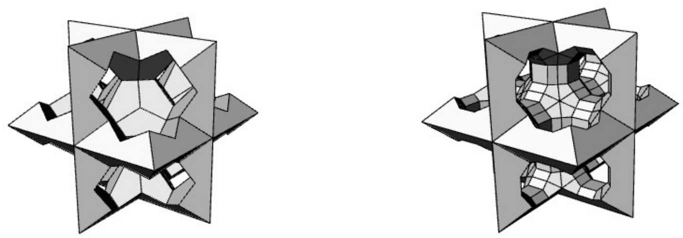

Figure 1: *Twelve toroids created from tetrahedra divided by a polyhedral surface (a) (6^4), (b) (4^3,6).*

Diamond

When the Conway tetrahedra created from the diamond lattice are divided by planes the bisecting surface derives from a space-filling of rhombic dodecahedra, created by removing half of the faces from each of them to leave hexagonal rings (Figure 2). Its dual, (4,6,4,6), is derived from a space-filling of truncated octahedra by omitting half the hexagonal faces (Figure 3a,b). When the regular infinite polyhedron (6^6) is the bisecting surface its faces extend into neighbouring tetrahedra (Figure 3c).

Figure 2: *(a) Part of the dual (4,6,4,6) polyhedron, the bisecting surface when tetrahedra from the diamond lattice are divided by planes.(b)The author building a model of it at the ATM conference, 1997.*

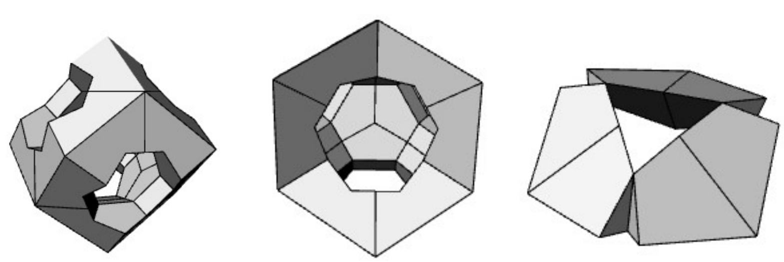

Figure 3: *(a) Four toroids from tetrahedra divided by the surface (4,6,4,6). (b) A different view with one of the toroids removed. (c) A toroid from tetrahedra divided by the surface (6^6).*

Triamond

The other lattice considered by Séquin has three edges meeting at a vertex and the minimal circuits have ten edges, so its Schläfli symbol is (10,3). It was described by A.F.Wells, where it is example (a) in his figure 37 [16], so it is known as (10,3)-a. It has been rediscovered several times and is also known as the Laves graph, K4 [12] and the triamond [1]. It is chiral, with the pair of dual lattices mirror images. The corresponding minimal surface that divides space into indirectly congruent halves is Schoen's gyroid [9], but there is no infinite polyhedron analogous to the previous examples. The tetrahedra constructed in the usual way are not all identical, so there are different patches created by planar division: in each toroid there are two squares, and eight 60° rhombi, which are coplanar in sixes when the complete surface is assembled (Figure 4a). The resulting infinite polyhedron is called the multiplied snub cube [1], and has been used as a pattern to make a model [15]. It is usual to interpret the six rhombi as a hexagon and six coplanar triangles, so that all the faces are regular polygons, but then the faces of the dual polyhedron are not planar. If the complete rhombi are seen as the faces as in Figure 4a then the dual faces are planar (Figure 4b): regular hexagons and 80.406° rhombi. I believe this polyhedron has not been described before.

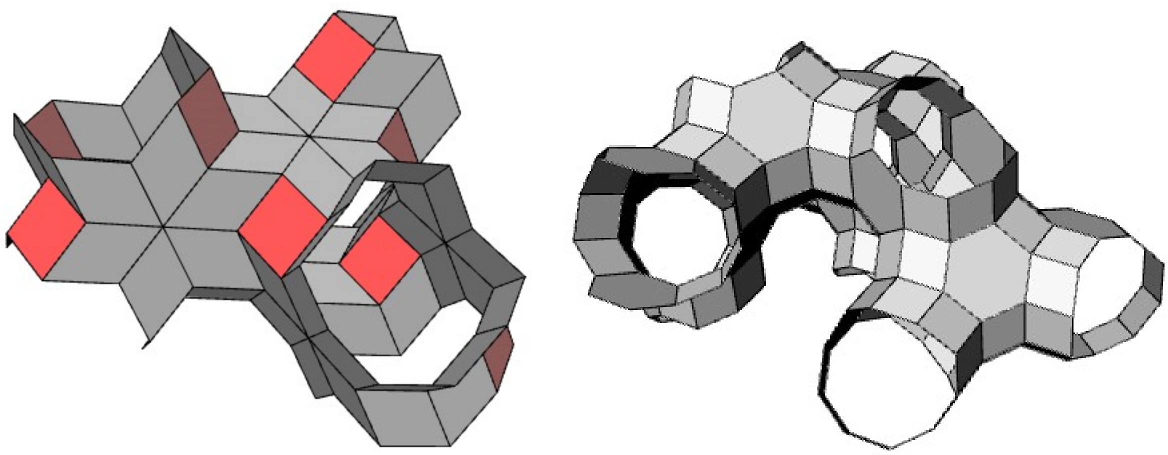

Figure 4: *(a) An infinite polyhedron approximating to the gyroid. (b) Its dual polyhedron.*

Stella Octangula

There is one other self-dual lattice with cubic symmetry. It has not received much attention, and I am not aware that it has been given a name. If it is mentioned at all it is usually for completeness. It is not regular because there are two types of node: one has twelve edges, the other has four. The easiest way to understand it is to think about a packing of cubes, each inscribed with a stella octangula (Figure 5). Their edges form the lattice. The 4-valent nodes are at the centres of the cube faces; the 12-valent nodes are where the vertices of the cubes meet.

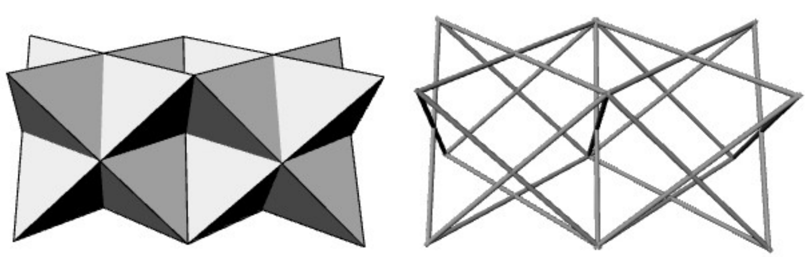

Figure 5: *(a) Two stellae octangulae packed together, and (b) their edges.*

The minimal surface that divides the self-dual lattice pair was discovered by the Finnish mathematician, Neovius, in 1883. There is only one bisecting surface with regular polygonal faces, (4,8,6,8). It can be created from the same packing of Archimedean polyhedra as (4^3,6), this time by omitting all the squares from the truncated cuboctahedra and half the ones from the octagonal prisms.

Toroids produced by planar division of tetrahedra are shown in Figure 6 along with the corresponding pieces from the bisecting surface. It is the dual of (4,8,6,8) and has faces that are 60° rhombi. It does not appear in any of the well-known descriptions of infinite polyhedra [6, 14].

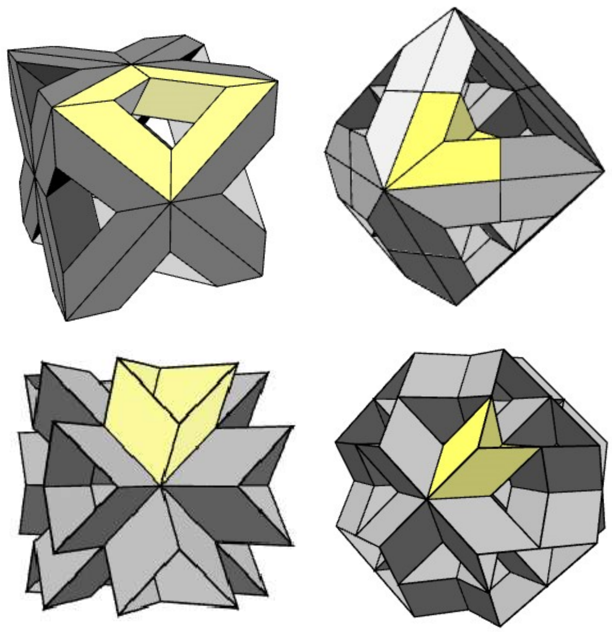

Figure 6: *(a) Twelve toroids arranged to emphasize the 4-valent nodes. (b) 24 toroids arranged differently. (c) The piece of the dual (4,8,6,8) polyhedron included in (a). (d) The piece included in (b).*

Circular Rings

Rinus Roelofs has described two 3-D structures with interlinked circular rings that derive from the cubic and diamond lattices [7]. He used annular rings but maybe it is more natural to use tori that can fit inside the toroids in the previous space-filling arrangements. The cubic structure seems fairly obvious, the one from the diamond lattice maybe less so (Figure 7) because the minimal circuit of the diamond lattice is non-planar, and the toroid derived from it has a corresponding wavy shape (Figure 3a,b), but it is still possible to fit a torus inside it. The minimal circuit of the triamond lattice is also non-planar but it is not possible to fit a torus inside the corresponding toroid.

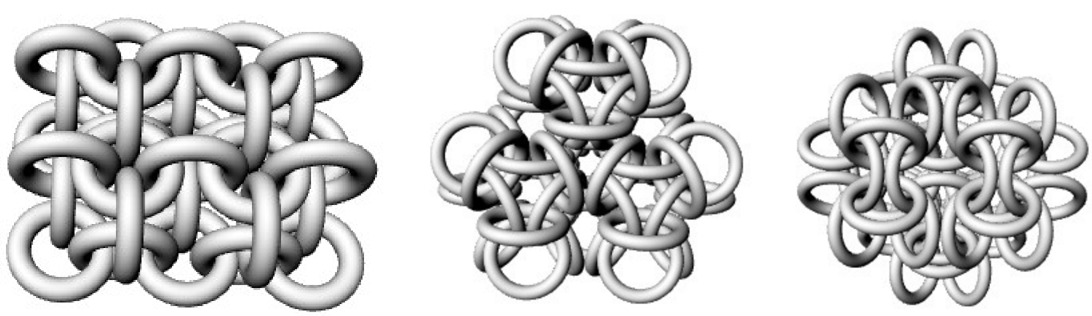

Figure 7: *(a) Linked rings in the cubic lattice. (b) Linked rings in the diamond lattice seen along a threefold axis of symmetry. (c) Linked rings in the diamond lattice seen along a twofold axis of symmetry.*

The tori derived from the stella octangula lattice lie in the faces of rhombic dodecahedra (Figure 8). Carlo Séquin has described a similar link by thinking about cubic symmetry [11]. It has two different kinds of torus, which can be derived from the dual pair of lattices, one consisting of the edges of a space-filling of tetrahedra and octahedra, the other the edges of a space-filling of rhombic dodecahedra.

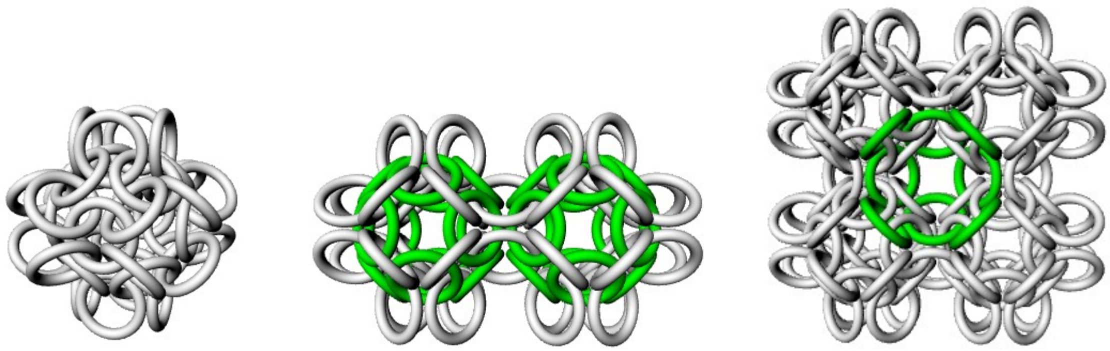

Figure 8: *(a) A unit of linked rings in the stella octangula lattice. (b) Two connected units showing the rhombic dodecahedra. (c) Four units coloured to show a rhombic dodecahedron from the dual lattice.*

Regular Polylinks

Alan Holden has described what he calls polylinks, which consist of polygonal rings arranged with polyhedral symmetry [5] and it seems natural to extend the idea to infinite polyhedra, although nobody seems to have done this before. The resulting arrangements are generally quite different from those considered so far, which all consist of a pair of disjoint sets of rings divided by a triply periodic surface. Nevertheless they are still triply periodic, and if the base infinite polyhedron is monohedral the polygonal rings are congruent. Unlike Holden's finite examples they must lie in the same plane as the polyhedron faces if they are to interlink.

Truncated Octahedron

The simplest case to consider is (6^4). George Hart has described a polylink having eight interlinked hexagons [4]. He explains it as being based on the octahedron, although it would be more accurate to say a truncated octahedron minus the squares, which is exactly what is needed for (6^4). It is not too difficult to arrange things so that the rings interlink without colliding with each other (Figure 9a). In fact it is even possible to include squares (Figure 9b), so that a polylink based on (4,6,4,6) is possible, but of course it has two kinds of polygonal ring.

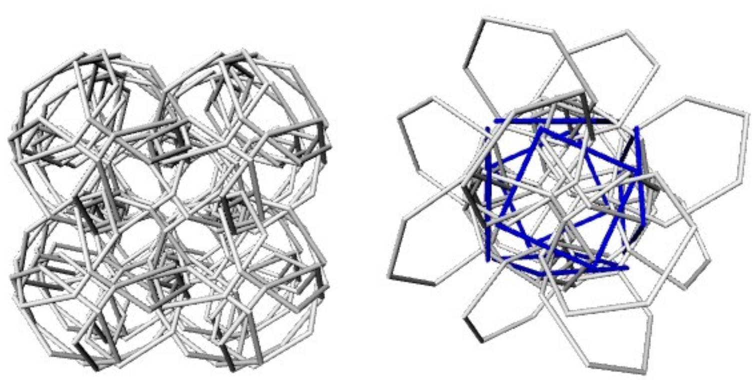

Figure 9: *(a) The polylink based on (6^4). (b) The same polylink can be adjusted to have room for squares.*

Four hexagons interlink at the vertices of (6^4). If only half of the corners in each face are used then triangles result, and they can link across a polyhedral vertex to the opposite triangle because the corners that would otherwise have got in the way are not there. In fact this creates two disjoint polylinks that never connect. If lattices are constructed with nodes at the centroids of the triangles and edges wherever they interlink, the dual triamond pair is produced (Figure 10). Tom and Koos Verhoeff named a structure like this "Bamboozle" [13]. The rings in these polylinks cannot be separated into two sets by a surface.

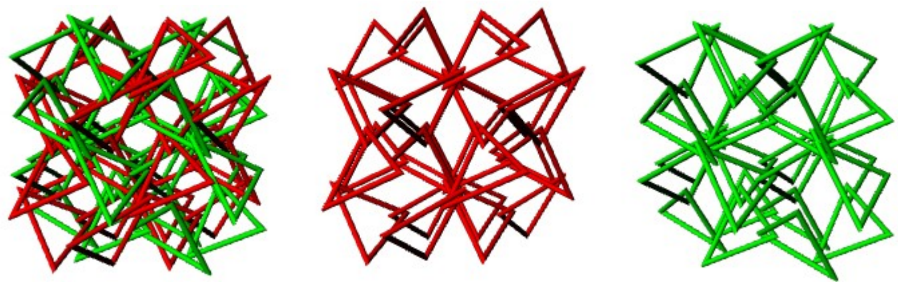

Figure 10: *(a) Linked triangles using half the corners of the faces of (6^4). Triangles have the same colour if they are linked. (b) Triangles of one colour. (c) Triangles of the other colour.*

The triamond has planar 3-valent nodes and analogous links exist associated with lattices having planar 4-valent and 6-valent nodes, although they do not count as polylinks since the rings do not derive from the faces of infinite polyhedra. The edges of the 4-valent lattice are those of a space-filling of stellated rhombic dodecahedra, or alternatively saddle octahedra. The dual lattice is called body centred cubic, and has nodes connected like the centre of a cube to its vertices. The associated link is like Figure 7a with square rings rather than tori.
The edges of the 6-valent lattice are those of a space-filling of a less well-known saddle polyhedron called the tetrahedral decahedron by Pearce [6]. The associated link is like Figure 7b,c with hexagonal rings rather than tori.

Cube

Creating polylinks based on the faces of (4^6) is almost as straightforward (Figure 11). It is slightly more difficult to find than the finite cubic polylink, but only because the faces of the infinite polyhedron are coplanar in pairs at each vertex. This puts a limit on the size of the square rings depending on the amount they are turned relative to the polyhedral faces.

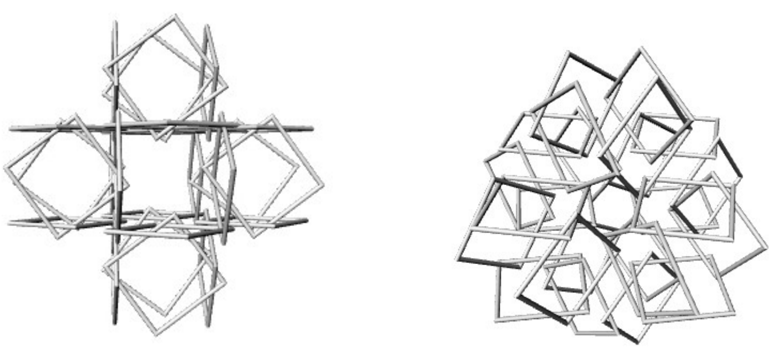

Figure 11: *Part of the polylink based on (4^6) (a) seen along a fourfold axis of symmetry, (b) along a threefold axis of symmetry.*

Truncated Tetrahedron

There are two narrow ranges of parameters that work to give polylinks based on the faces of (6^6). In both cases the rings need to be very thin, and the main difference between the two is the amount of interlinking. In the simpler case each ring links only with rings from the same truncated tetrahedron: six rather than three since the polyhedron bisects space into congruent halves, so there is a truncated tetrahedron on both sides of a face. In the other there are two more linked rings at each corner: 18 in all (Figures 12, 13).

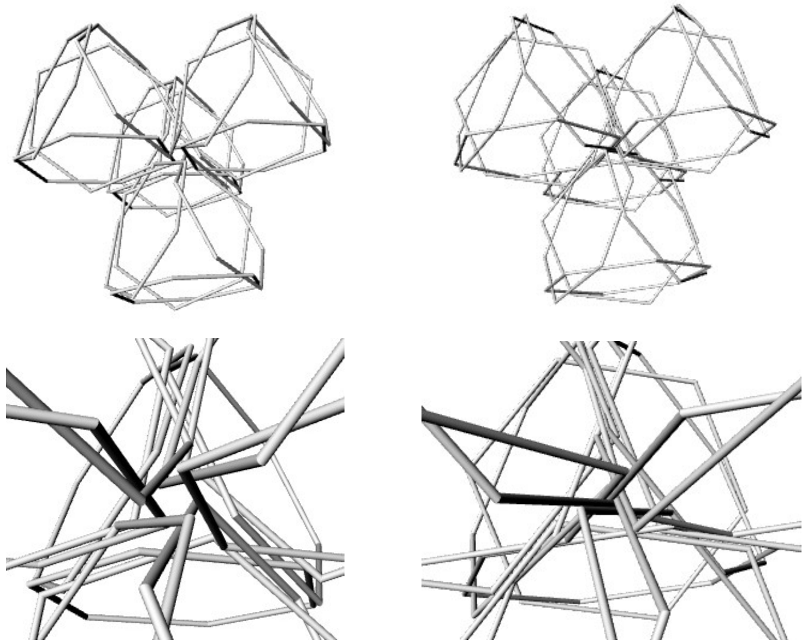

Figure 12: *The two (6^6) polylinks looking along a threefold axis with enlargements.*

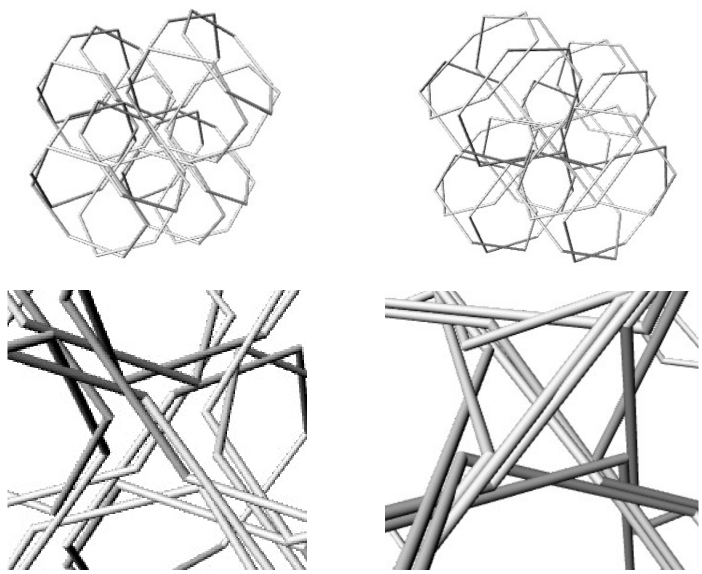

Figure 13: *The two (6^6) polylinks looking along a twofold axis with enlargements.*

Further Possibilities

The conditions set out in the introduction are quite rigorous: planar rings that are regular and equivalent under the cubic symmetry of an infinite link. Relaxing the planar requirement on circular rings, as Frank Farris has done in designing chain mail [2] allows more possibilities, so that it becomes possible to use toroids from the triamond lattice. The problems with polylinks can be easily sidestepped, although allowing non-planar rings seems contrary to the spirit of Holden's original concept. A polylink having two kinds of ring has already been mentioned in passing (Figure 9b) and clearly any infinite polyhedron with regular faces might provide another, but Coxeter's regular sponges are the only completely regular ones, and they have been dealt with already.

There are no more self-dual lattices with cubic symmetry, but there are two more dual pairs [14]: the body centred cubic one and its dual with planar 4-valent nodes mentioned previously; and the edges of a space-filling of tetrahedra and octahedra, dual to the edges of a space-filling of rhombic dodecahedra, mentioned in connection with Séquin's link [11]. Conway's tetrahedral construction is guaranteed to give space-filling toroids, but there are two different kinds.

While artists such as John Robinson have created sculptures that are links, they are finite. The problem with using an infinite structure is that only a finite piece can be created. Tony Smith has explored fragments of infinite structures, but I am not aware of any sculptor who has considered using infinite links like this, although there are several who have worked with pieces of triply periodic minimal surfaces.

References

[1] J.H. Conway, H. Burgiel, C. Goodman-Strauss. *The Symmetries of Things.* A K Peters, 2008, p.351.

[2] F. Farris. "Wallpaper Patterns from Nonplanar Chain Mail Links." Proceedings of Bridges 2020: Mathematics, Art, Music, Architecture, Education, Culture, pp.183-190.
http://archive.bridgesmathart.org/2020/bridges2020-183.html

[3] P. Gailiunas. "Transforming Some Infinite Polyhedra." *Bridges: Mathematical Connections in Art, Music, and Science*, 2002, pp.189-195.
http://archive.bridgesmathart.org/2002/bridges2002-189.html

[4] G.W. Hart. "Orderly Tangles Revisited." *Renaissance Banff: Mathematics, Music, Art, Culture*, 2005, pp.449-456.
http://archive.bridgesmathart.org/2005/bridges2005-449.html

[5] A. Holden. Orderly Tangles: Cloverleafs, Gordian Knots, and Regular Polylinks, Columbia U., 1983.

[6] P. Pearce. *Structure in Nature is a Strategy for Design*, MIT, 1978.

[7] R. Roelofs. "Entwined Circular Rings." Bridges Donostia: Mathematics, Music, Art, Architecture, Culture, 2007, pp.81-90.
http://archive.bridgesmathart.org/2007/bridges2007-81.html

[8] W.W. Rouse Ball, H.S.M. Coxeter. *Mathematical Recreations and Essays*, Twelfth Edition, University of Toronto Press, 1974. p.152-3.

[9] A.H. Schoen. *Infinite Periodic Minimal Surfaces Without Self-intersections*. NASA TN D. National Aeronautics and Space Administration, 1970.

[10] C.H. Séquin. "Intricate Isohedral Tilings of 3D Euclidean Space." *Bridges Leeuwarden: Mathematics, Music, Art, Architecture, Culture,* 2008, pp.139-148.
http://archive.bridgesmathart.org/2008/bridges2008-139.html

[11] C.H. Séquin. "Polyhedral-Edge Knots." Proceedings of Bridges 2021: Mathematics, Art, Music, Architecture, Culture, pp.63-70.
http://archive.bridgesmathart.org/2021/bridges2021-63.html

[12] T. Sunada. "Crystals that Nature Might Miss Creating." *Notices of the American Mathematical Society*, 55(2), 2008, pp.208–215.

[13] K. Verhoeff, T. Verhoeff. "Folded Strips of Rhombuses and a Plea for the √2:1 Rhombus." *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*, pp.71-78.
https://archive.bridgesmathart.org/2013/bridges2013-71.html

[14] A.M. Wachman, M. Burt, M. Kleinman. *Infinite Polyhedra*. Technion, 1974 second edition 2005.

[15] M. Weber. Make your own gyroid, 2018.
https://minimalsurfaces.blog/2018/12/23/make-your-own-gyroid/

[16] A.F. Wells. *The Third Dimension in Chemistry*. Oxford University Press, 1956.