# Geoweaving: Fold-Up Baskets from Dessins d'Enfants

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## Abstract

Dessins d'enfants, also called hypermaps, are closely associated with ramified coverings of the Riemann sphere—what a layperson may prefer to think of as seamless wallpaperings of other closed surfaces with copies of the Earth's surface (Earth serving in lieu of the Riemann sphere.) Dessins d'enfants have been a focus of interest in recent years in fields as challenging as number theory and quantum gravity. Let's add basketmaking to the list. I show that Strobl's clever knotology weaving technique—which realizes curved surfaces in plain tabby weave by using straight (but folded) weaving elements—suffices to make models of the Riemann surfaces encoded by dessins d'enfants. These baskets can be folded up into a single stack of triangles—an echo of a Riemann surface being a ramified covering of the sphere. Plans and weaving instructions are provided for an example the reader can make.


## Introduction

As recently as the last decades of the 20th century researchers in many fields were working with the same mathematical object, calling it by different names, and largely unaware of each other's work. The book by Lando, Zvonkin, and Zagier [2] surveys an amazing breadth of appearances of *dessins d'enfants* in many fields of mathematics and science, but less known is their relevance to basket weaving. This paper shows how a connected line drawing (perhaps one drawn by a child, as Grothendieck's term suggests) can become, through a rule-based, post-production process, a basket woven with the straight, constant-width weaving strands of Heinz Strobl's knotology weaving technique [3]. Since dessins classify Riemann surfaces, the baskets they direct have unusual properties (Figure 1): seamlessly wallpapered by copies of the globe, they can fold up into a single copy of the globe (a hands-on model of a ramified covering of the sphere.) I provide plans and instructions for a simple basket of this type. Those intrigued by the connections of dessins to polynomials and the algebraic numbers will find [5] a good introduction with many illustrated examples.

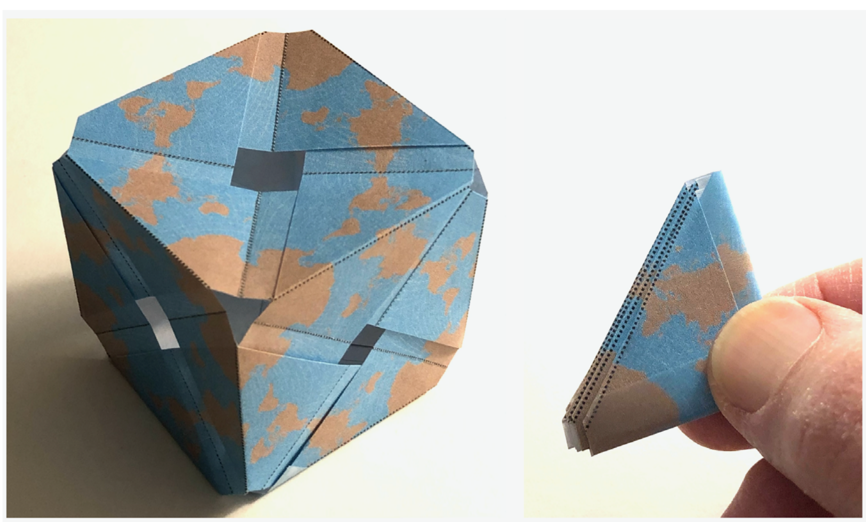

**Figure 1:** *Because dessins classify Riemann surfaces, the baskets they direct can be covered seamlessly by copies of the globe, and are able to fold into a single copy of the globe.*

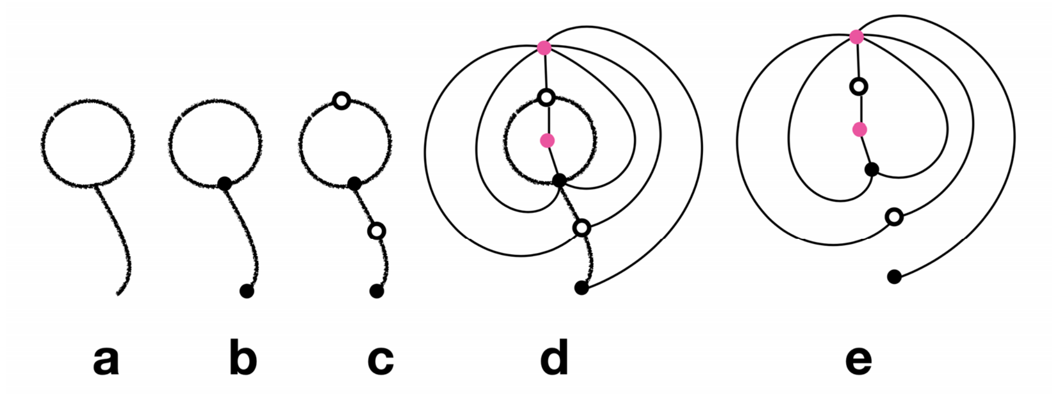


**Figure 2:** *From a child's drawing to a quadrangulation.*

## From a Drawing to a Canonical Quadrangulation

Dessins have deep connections in mathematics—number theory in particular—areas we will not need to touch upon. For us, dessins will just be a special sort of graph (a bicolored graph) drawn on a spherical surface.[1] Let's look at the pipeline of how a connected, line drawing on the sphere can become a dessin and then a basket (Figure 2.) The artist draws a line drawing (Figure 2a.) The artist is drawing on a sheet of paper, but the drawing will be understood mathematically as having been drawn on a sphere. The one rule is that all the lines must connect.

The drawing is completed. Now begins the rule-based, post-production process: add a black vertex at each location where lines cross or end (Figure 2b.) We now have a graph; in fact, a graph drawing properly embedded on the sphere. Since we are on a closed surface, the white space surrounding the drawing is also a face. For example, in Figure 2b, the embedded graph has two faces: a 1-gon and a 3-gon. Insert a white vertex (unfilled circle) in the middle of each graph edge (Figure 2c.) Importantly, this guarantees that each face now has an even number of sides. At this point we have a proper vertex 2-coloring of a bipartite graph, a *bicolored graph.* (A graph is said to be bipartite if it admits a proper 2-coloring of its vertices; in other words, the vertices are able to be colored either black or white such that no edge joins vertices of the same color.) By inserting a white vertex in the middle of each edge, we have converted a graph (that perhaps was not bipartite) into one that is bipartite; and we have singled out one of its (only two) possible bicolorings. Such a bicolored, embedded graph—when considered up to topological equivalence—is known as the *Walsh representation* of a hypermap. 'Hypermap' is just another word for dessin: the drawing has become a dessin.[2]

Now we move toward basket weaving. Add a pink vertex in the middle of each face (Figure 2d,) and add new edges connecting the pink vertex to the black and white vertices incident to that face, thereby creating a mesh of topological triangles. Every vertex now has an even number of edges (i.e., we now have an Eulerian graph) and therefore—triangles being easier to count—each vertex also has an even number of triangles around it. In addition, we are in possession of a proper vertex 3-coloring for this Eulerian triangulation. What we now have is known as the *canonical triangulation* of the dessin. We won't actually put the following colors on the edges, but we could hypothetically assign to each edge in the triangulation the color of the opposite vertex (such color being the same on either side of the edge.) With that understanding: delete the 'pink edges' (i.e., delete the original artist-drawn edges.) We now have the *canonical quadrangulation* of the dessin (Figure 2e) where every face is a quad, and every quad has a pair of opposite corners where one vertex

[1]The procedure works out just fine on higher topology orientable surfaces, but the drawing will need to divide the surface into simply connected regions—a lot to ask of our young artist.

[2]Dessins are not limited to 2-valent white vertices—that is just a result of starting from an ordinary graph. More generally, one finds a bipartite graph and chooses a bicoloring.

is black and the other white, call these the *polar colors.* In the other pair of opposite corners both vertices are pink, call pink the *non-polar* color.

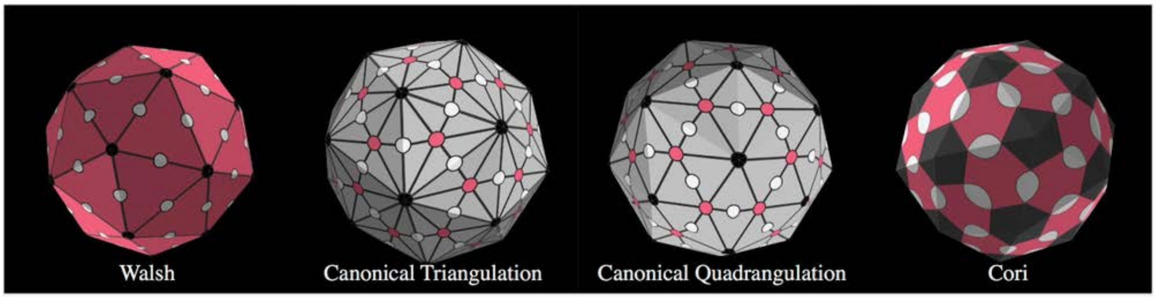


**Figure 3:** *Four classical representations of the same dessin. The Cori representation, which is dual to the canonical quadrangulation, can be read as a basket weaving diagram. Imagine wire strands that follow the boundaries between colors. They pass over and under in a way that gives the color-coded weave openings the correct helical handedness: the polar colors (black and white) code one handedness, the non-polar color (pink) codes the other.*

## The Canonical Quadrangulation Describes a Basket

If we choose to see the faces of the canonical quadranglation as indicative of the quadrangles of exposed strand seen in densely-woven tabby weave, and the polar vertices (respectively, non-polar vertices) as indicative of weave openings of right-handed (respectively, left-handed) helicity [4], then the vertex 3-colored canonical quadrangulation already specifies a particular basket, one woven in a particular handedness. Easier for the weaver to read and draw is the dual graph, the *Cori representation* (Figure 3), a 4-regular graph with a proper face 2-coloring (i.e., polar versus non-polar color.) The Cori representation (sometimes referred to as the *medial* or the *ambo* of the Walsh representation) is indicative of the appearance of sparse weaving (imagine wires tracing the boundaries between colors.)

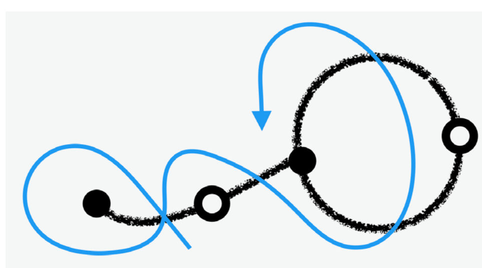

**Figure 4:** *The paths of weaving strands can be found most easily by following a 'refractive path': crossing obliquely over the middle of an edge and then steering for the next mid-edge.*

I draw the medial of the Walsh representation (Figure 4) by following what I call 'refractive paths' around the dessin with colored pencils. The technique is to start out by crossing obliquely the middle of an edge, and then steering toward the middle of the next edge. If the cycle closes before crossing every edge of the graph twice, start a another cycle in a different color. For example, the graph in Figure 4 will need one more cycle to double-cover all the edges. The medial graph quickly clarifies the important details of the weave.

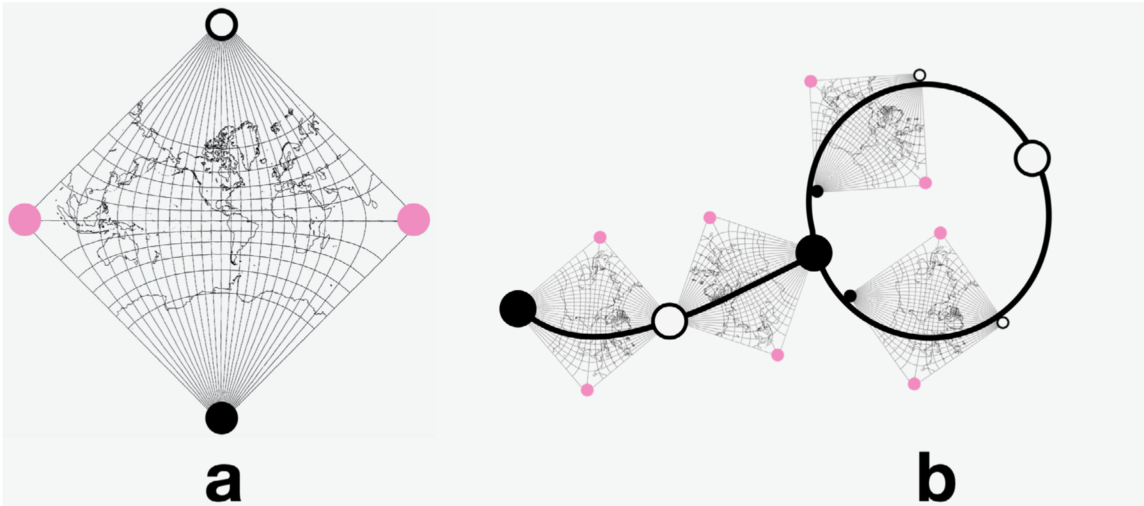


**Figure 5:** *The Adams 'World in a Square II' map tile (a), conformal except at its corners, can be tiled into a seamless, doubly-periodic version of the Earth's surface. Any bicolored map on an oriented surface (b) shows a way to tile that surface with Adams tiles: each map edge gets its own tile; black/white corners meet over map vertices of like color, pink corners meet over the center of their respective map faces. Each black-to-white edge of the map corresponds to a south-to-north traverse of the Prime Meridian on its tile. The tiles can be folded along their Prime Meridian, and, as here, that is commonly necessary.*

## The Adams 'World in a Square II' map projection

In 1929, Oscar Sherman Adams, chief mathematician at the United States Coast and Geodetic Survey, published a new map projection [1] (Figure 5a) dubbed *World in a Square II.* Adams' projection has the novel property that it tiles into a seamless, doubly-periodic version of the Earth's surface. The projection is conformal (i.e., the map distortion consists of local variations in scaling: small circles remain circles, angles are preserved)—conformal everywhere except for the four corners[3]. That is to say, it is non-conformal at just three locations on the planet: the South Pole, the North Pole, and the 'Mid-Pacific Point' where the Equator crosses the Anti-Meridian (180°W.) In each tile, at opposing corners, are the North and South Poles; at the other pair of opposing corners are two complementary copies of the Mid-Pacific Point (each copy carries only half of the Mid-Pacific Point's environs.) Sound familiar? Clearly the coloring of the canonical quadrangulation tells us exactly how to place Adams tiles on a basket—we only need to fix a canonical mapping of South/North to Black/White; the Mid-Pacific Point must map to Pink. Figure 5b shows a shortcut from a bicolored graph drawing (as in Figure 2c) to a canonical quadrangulation already tiled.

## Strobl's Knotology Weaving

In the 1990's Heinz Strobl [3] invented a new type of weaving (Figure 6) that he dubbed knotology. Knotology reconciles the two fundamental ways to weave: tabby weave and kagome. The weaving elements that achieve this alchemy are straight, standard-width, rectangular strips that are creased in an unvarying sequence of right triangles. Remarkably, nearly 100 percent fabric coverage is achieved (the characteristic hexagonal 'eyes' of the kagome weave are closed up.) Though knotology is limited to thin materials, like paper or sheet metal, that can sustain a sharp bend, it achieves surprising versatility from its standard parts.

[3]The Adams tile manifestly does not preserve angles at its corners: the North and South corners have 360° represented in 90°, and each of the Mid-Pacific corners has 180° represented in 90°.

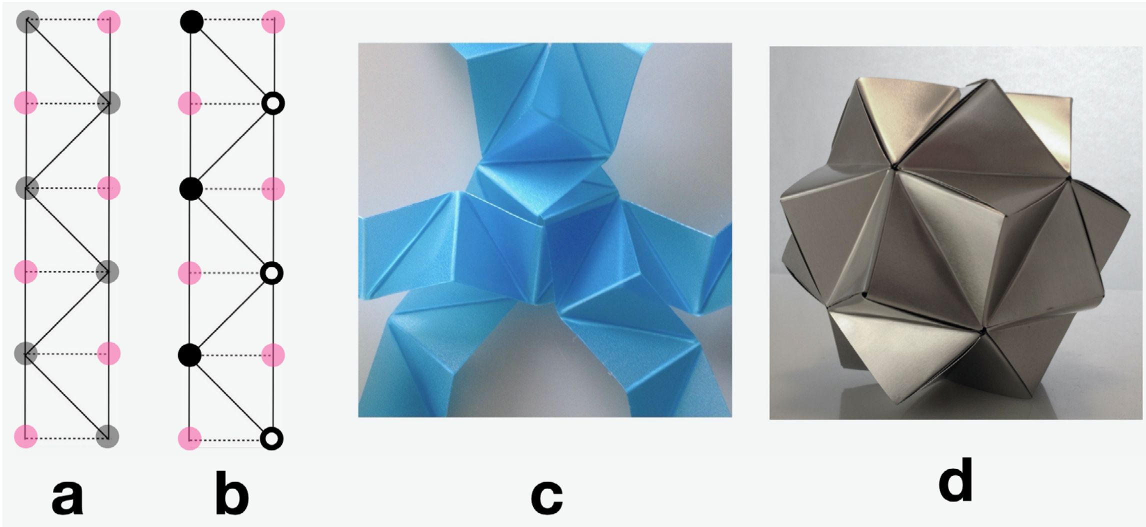


**Figure 6:** *Knotology weaving: (a) a knotology weaving strip with color-coded corner matchings; (b) the corner matchings needed after a weaving strip is decorated with Adams tiles; (c) three interwoven knotology strips sharing a corner; (d) a metal sculpture in knotology.*

Knotology weaving requires the corner matchings indicated in Figure 6a; by comparison, the corner matchings required by a strip of Adams tiles are indicated in Figure 6b. What were uniformly gray corners in knotology weaving, now alternate black/white in a strip of Adams tiles. Some of the symmetry and versatility of knotology is sacrificed when the weaving strands are decorated with Adams tiles; in particular, the diagonal folds—the Prime Meridians—must trace a *bipartite* graph. Notice that the diagonal folds of the knotology strand must coincide with the Prime Meridian on the tiles if we hope to be able to weave a basket that wears a single copy of the Earth, since that is the only place where "World in a Square II" can fold to properly pair up its edges.

## Instructions for the Preliminary Exercise

At this point I believe you will not find it difficult to follow the instructions in the next section to make the model in Figure 1; however, it may be advisable to start with something more minimal and more representative of what you will encounter if you start from an arbitrary drawing. For this exercise you will need to print a copy of Figure 7 and cut out three geoweaving strands, then follow the folding instructions in the first paragraph of the next section. For the exercise we will weave the basket in Figure 5b. It can be clarifying to make a model of a surface before attempting to weave it. Figure 5b indicates that we will need four tiles to make a model of the surface. (In cutting geoweaving strands to length, notice that there are two different places where a cut can be made: cuts are either *southern* or *northern* according to whether they cut along the southern or northern portion of the Anti-Meridian—think ahead if there is a particular place where you want the cut and splice to lie.) Cut a strand four tiles long starting from a southern cut. We will be working left-to-right from the 1-valent black vertex on the left of the Figure 5b (recall that Black = South Pole; White = North Pole; Pink = Mid-Pacific). Form the Black-White-Black vertex sequence by folding two successive tiles along Prime Meridians. These two folds form a shape like a square envelope cut off along its diagonal (Figure 8a.) The last two tiles in the strip fold back along an Anti-Meridian to form a 2-valent Pink vertex encircled by two Prime Meridians. Double check that the valences and connectivity of all the vertices agree with the figure, and then use clear tape to close up the seams (Figure 8b.)

It helps to trace the paths of the weavers on the model by wrapping it with masking tape (Figure 8c.)

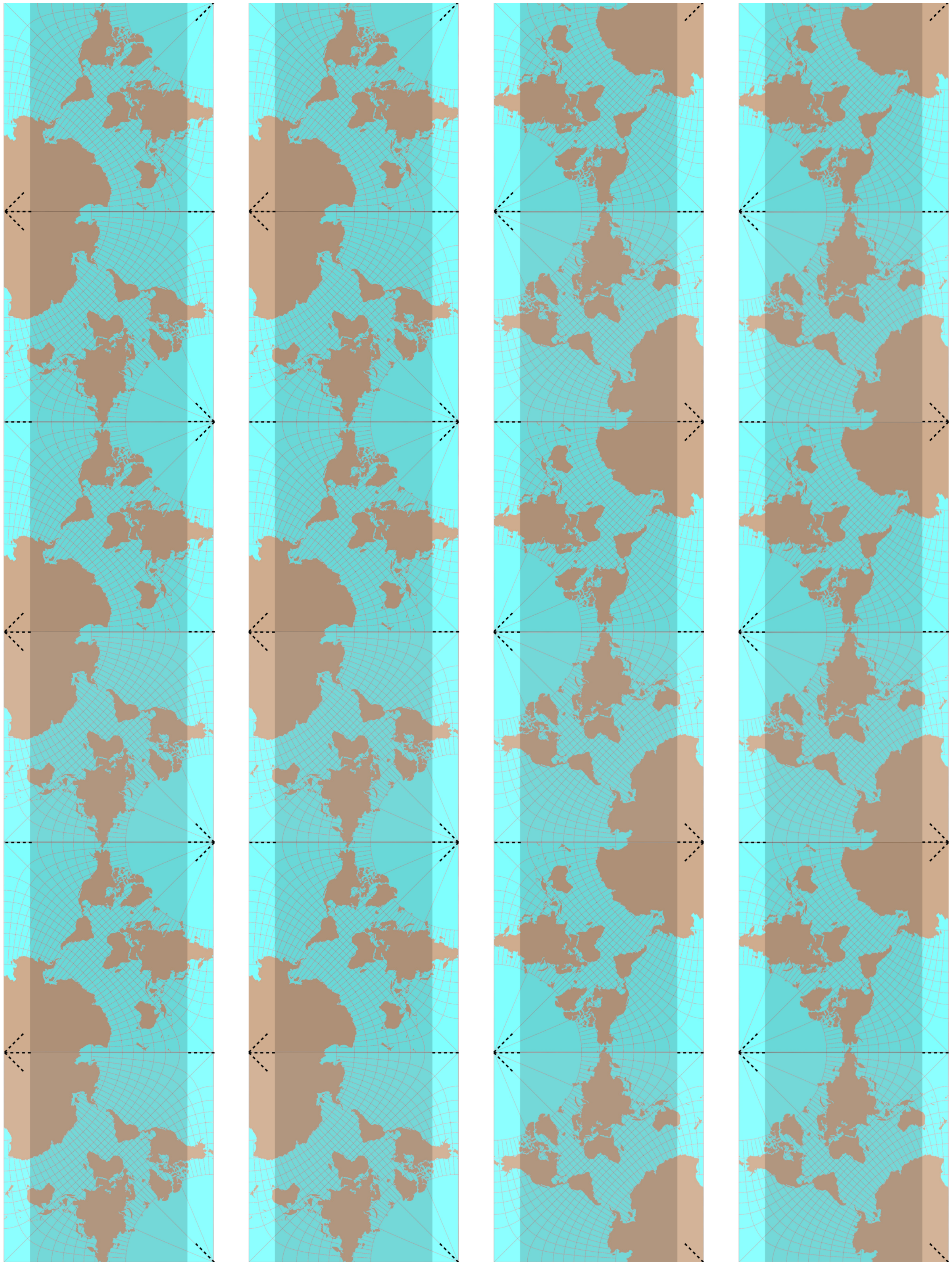

**Figure 7:** *Geoweaving strands for the model. (Fold lines are indicated only at the edges.)*

You should find that you need one 6-tile strand and one 2-tile strand to do the weaving (just as predicted in Figure 4.) Because weaving needs some maneuvering room, trim the light-colored margins off the two remaining strands that you folded for this exercise. It is best to start weaving at an intersection to give a clear reference point for the over-and-under rhythm. Cut a two-tile strip, and splice it into a loop with clear tape. Refer to your model to see how you want the map of the globe to be oriented where the two loops cross. The longer loop is six tiles long so it does not need cutting to length. Slide the two-tile loop over the six-tile strip to a position where the globe orientations agree with each other and the crossing in the model. Starting from there, fold the 6-tile loop to mimic the model while maintaining an under-and-over rhythm. The end of this strip will land next to where it started, allowing it to be spliced with a piece of tape.

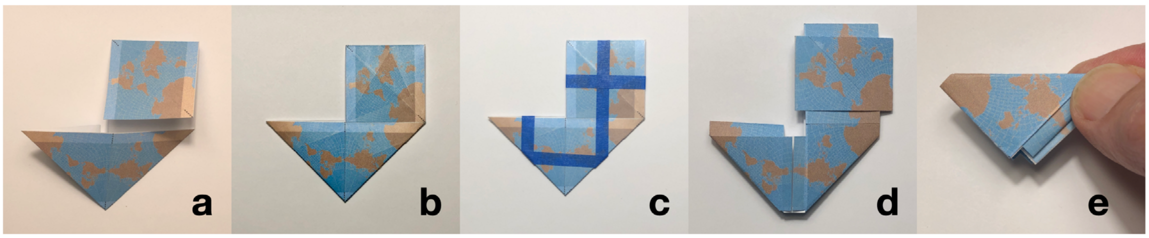


**Figure 8:** *Steps in the preliminary exercise.*

## Instructions for Weaving the Model

Print and cut out the four geoweaving strands in Figure 5. (Ignore the shading when you cut; the unshaded portions will be trimmed off later.) Fully crease along the transverse folds suggested by the markings on the edges. Then fully crease along the diagonal folds suggested by the markings on the edges. Reverse all the folds completely (i.e., mountain folds become valley folds) so that the hinges are fully broken in. At the end you will want each strand to be an accordion-folded stack with the diagonal folds appearing as mountain folds when seen from the printed side.

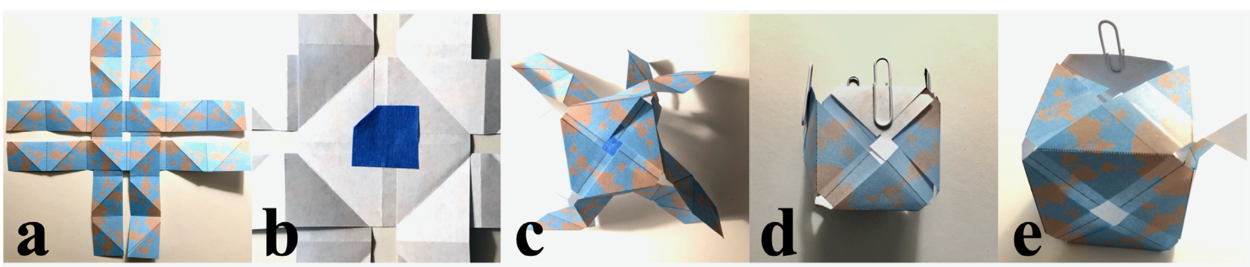


**Figure 9:** *Assembly steps for the model.*

Stretch out the strands and trim off the light-shaded margins that bear the fold indications. Narrowing the strands in this way loosens the weaving enough to accommodate the basket's conformation changes. The strands are of two types, but you can approach the first crossing as just an easy jigsaw puzzle. Find the center fold of each strand; it has a portion of the Mid-Pacific. Put together a smooth, 2-period copy of the Mid-Pacific, by assembling those four pieces. Put the four strands in a proper over-and-under weave (Figure 6a.) Hold the assembly together while you flip it over to the unprinted side. Tease the assembly back into proper order, making the diagonal creases coincide, above and below. There will be an open square in the center due to the trimming. When everything is aligned, cut a piece of removable tape about 2 cm square; double one of its corners back onto itself to make it easier to remove; adhere it to your skin and peel it off a few times making it easier to remove from paper without damage. Place this square of tape over the central hole to temporarily keep the weaving together (Figure 6b.) Flip the work over. You will see a diamond shape delineated by four diagonal folds, with a 'V' of strands coming from each of the four edges. Fold the 'V's downward so that the diamond shape of the bottom is clearly expressed (Figure 6c.) You can now weave each side of the box in turn: involved are the 'V' and a strand each from the neighboring sides. The over-and-under rule is now your friend, showing you the way. As each side of the basket is completed, fold back along the diagonal folds that delimit the top of the side, exposing the white underside of of a pair strands. These folded-back strands can now be secured at the top with a paper clip (Figure 6d.) When all four sides are clipped, it is time to peel that piece of removable tape off the bottom. The closure at the top of the basket requires splicing strand ends together with transparent tape like old-time audio editing. Start by removing two clips that share a corner. It will be obvious which pair of strand-ends want to mate up. Place a short length of transparent tape over one strand end, then position the other under the tape. The tape can overhang the sides of the splice because it will be easy to trim it perfectly flush with scissors once the splice is made. Work around the cube in the direction that lets you work without threading strands underneath (Figure 6e.) By the time you reach the last splice, the basket will be robust enough that you can reach in with your fingers and grab the strand that needs to come up from underneath.

The first change of conformation is the hardest. The mechanism is to fold both top and bottom inward hinging along face diagonals of the cube, each perpendicular to the other. You may want to practice this move in advance when the sides are clipped and the top of the basket is still open.

## Summary and Conclusions

I have shown that the popular topic of dessins d'enfants applies directly to the weaving of baskets, endowing such baskets, in some measure, with the wonderful properties of Riemann surfaces. I hope this can lead to some interesting learning by playing.

## Acknowledgment

I am grateful for the insightful suggestions made by the anonymous reviewers of this paper.

## References

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