Cutting and Sewing Riemann Surfaces in Mathematics, Physics and Clay

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Abstract

A series of ceramic artworks are presented, inspired by the author's research connecting theoretical physics to the beautiful theory of Riemann surfaces. More specifically the research is related to the classification of curves on the surfaces based on a description of them as built from basic building blocks known as "pairs of pants". The relevant background on this mathematics of these two dimensional spaces is outlined, some of the artistic process is explained: Both the conceptual ideas and their implementation. Many photos of the ceramics are included to illustrate this and the connected physics problem is briefly mentioned.

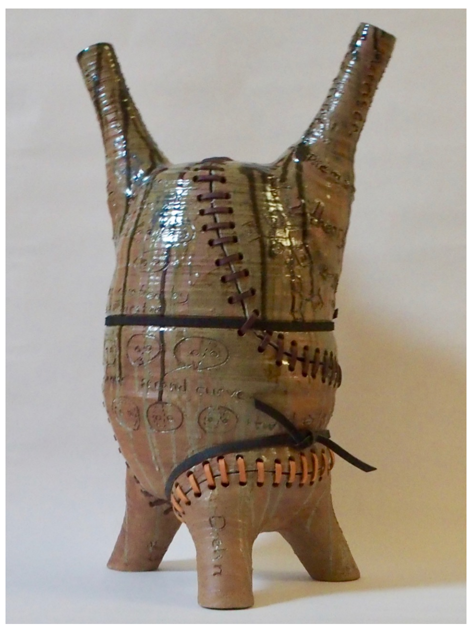


Figure 1: *"Sewing-7". Thrown and assembled stoneware with iron oxide wash, partial glazing and leather straps,* $63 \times 38 \times 26$*cm. The 5-punctured sphere is assembled from three "pairs of pants".*

Introduction

My speciality as a theoretical physicist is in string theory and supersymmetric field theories; both very abstract topics with closer connections to pure mathematics than to any observable phenomena. As such it is hard to communicate my research to audiences outside my subfield, which is where my hobby of ceramics comes in. For every research project that I undertake, I make a series of ceramic vessels or sculptures whose forms are inspired by the research and which I inscribe with details of the calculations involved.

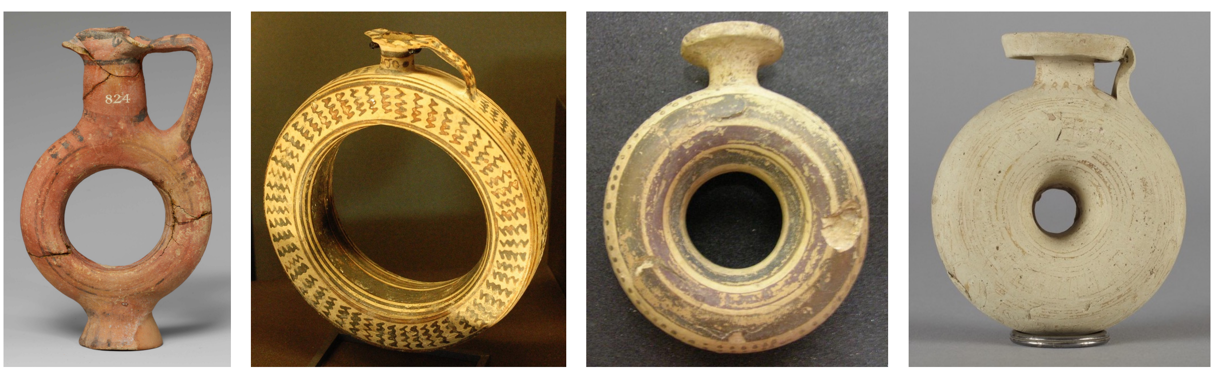
In this paper I tell the story of a series of artworks titled "Sewing", based on my paper [3] written with T. Okuda and D. R. Morrison. The paper deals with the problem of classifying line operators in certain supersymmetric field theories. I explain a bit about this question in the last section of this paper and focus mainly on the answer, which is easier: The classification exactly matches that of non-self-intersecting curves on two dimensional surfaces.

This classification was undertaken by Dehn [1] and independently by Thurston [10] (for a detailed discussion, see [8]) and their analysis influenced the design of my ceramics. It is based on the pair of pants decomposition of surfaces. As I illustrate below, oriented surfaces, possibly with punctures, can be described by sewing together basic building blocks which are known as pairs of pants (or three-punctured spheres). To be precise, this is true for surfaces of negative Euler characteristic and I do not distinguish between punctures and finite size holes which is unimportant when viewing the surfaces as topological rather than metric or complex spaces. All statements can be refined to the latter cases with a bit of care, see [3].

A curve on the surface may be parallel (homotopic) to one of the cuts where it is sewn, or it may intersect these cuts, extending from one pair of pants to another. The classification of curves (up to topological equivalence, that is smoothly deforming the contour), is then given by a set of integers representing the crossings and possible twists. The Dehn-Thurston parameters are a pair of integers for each "cut" i. One, p_i, representing the number of crossings and the other q_i the overall twist. $p_i \geq 0$ and if $p_i = 0$, then q_i represents the number of times the curve goes around the cut and in this case $q_i \geq 0$. In addition, for each pair of pants, the sum of the three p_i associated to the three legs has to be even, as we assume the curves are closed.

This classification (presented here very roughly) is quite intuitive and will hopefully become more so after reading the examples below. My purpose here is not to give a full account of this topic or the physics that is equivalent to it, but rather motivate the ceramic sculptures inspired by them.

The Once-Punctured Torus



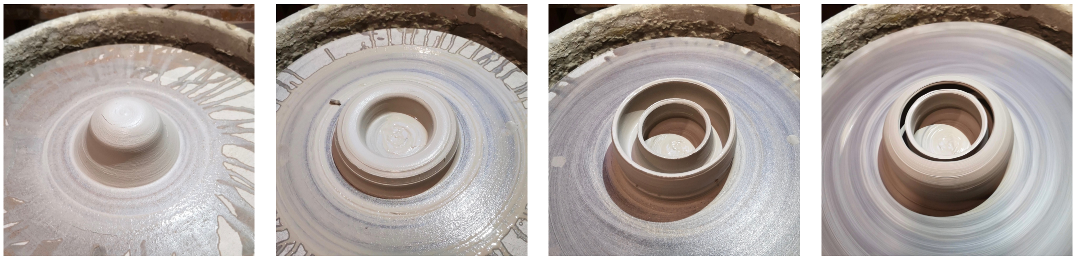
Cypriot, *850-600B.C.,* *h:*17.1*cm,* *the Met Museum [6].*	*Protocorinthian,* *675-650B.C.,* *h:*14.7*cm,* *Louvre Museum [5].*	*Early Corinthian,* *625-600B.C.,* $8.2 \times 6.8 \times 1.9$*cm,* *Ure Museum [12].*	*Late Corinthian,* *575-550B.C.,* $7.4 \times 6.3 \times 3.1$*cm,* *Princeton Museum [9].*

Figure 2: *Ancient ring aryballi.*

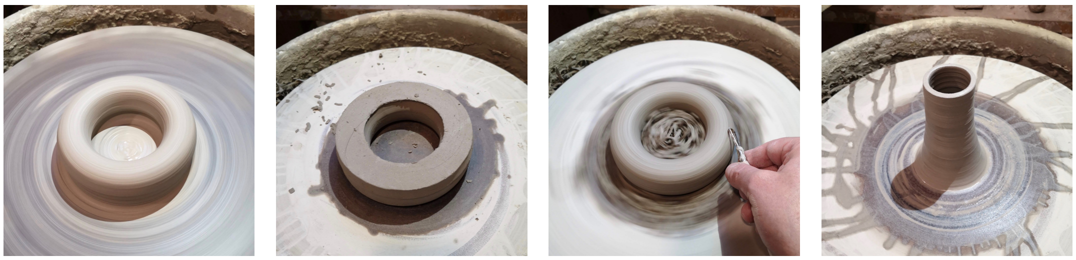
The simplest surfaces studied in [3] are the once-punctured torus and the four punctured-sphere. The former is made of a single pair of pants with two of the legs sewn together, making a torus and leaving one hole. The latter requires two pairs of pants sewn together along one of their holes.

There is a long tradition of of making toroidal jugs [11], see Figure 2 for some early examples.

Figure 3 illustrates the steps I use in making a torus on a modern potter's wheel. See the captions for details. The resulting piece is a torus on a stand where the latter represents a puncture in the torus. There



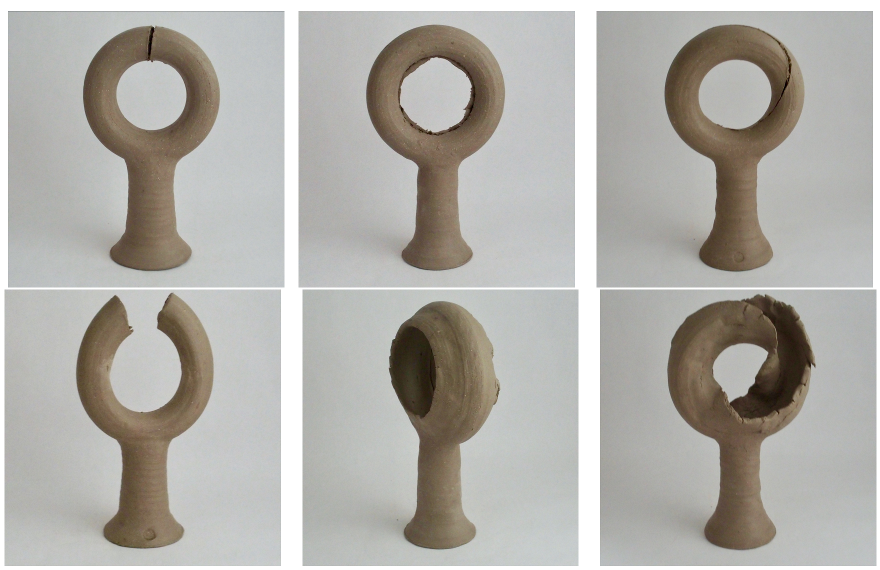
The clay is centered on the wheel. *It is opened up into a solid ring.* *Two walls are pulled up from the ring.* *The walls are bent towards each-other.*



They are fused to make a hollow torus. *Partially dried, the ring is placed upside down.* *Excess clay is removed to make it round.* *A tubular base is made and later joined.*

Figure 3: *Making a once-punctured torus on the wheel*

in fact has to be a puncture, as one cannot fire ceramic pieces with trapped air. As mentioned above, such a surface can be viewed as a single pair of pants with two legs sewn together. Conversely, we can cut the punctured torus to get the pair of pants, as I illustrate in Figure 4.



Cut along the short cycle. *Cut along the long cycle.* *Cut winding both cycles.*

Figure 4: *Cutting and deforming the once-punctured torus.*

There are an infinite numbers of ways to do that, but I made only three. It is easy to see that after the cut we have a surface with three boundaries, which is what is meant by a pair of pants. The third example in Figure 4 is a bit counter-intuitive, and does not look like normal trousers. rather two of the boundaries are circles which are linked with each other. If we were to trace these circles with yarn, the two loops would be interlocked. Still, as far as the intrinsic (two dimensional) properties of the surface is concerned, it is not really different than the other two examples, and this arises because of the way it is embedded in three dimensions.

Other pair of pants decompositions would get more and more complicated—the two circles would twist around each other multiple times.

The pieces in Figure 4 are for illustration purposes. For my sculptures, I do cut them, but aim to keep the original shape of the torus. I also pierced holes along the cut before firing the pieces. After the works come out of the kiln, I use yarn or leather straps to "sew" the pairs of pants back into once-punctured tori. The resulting completed pieces form the triptych "Sewing-1", shown in Figure 5.

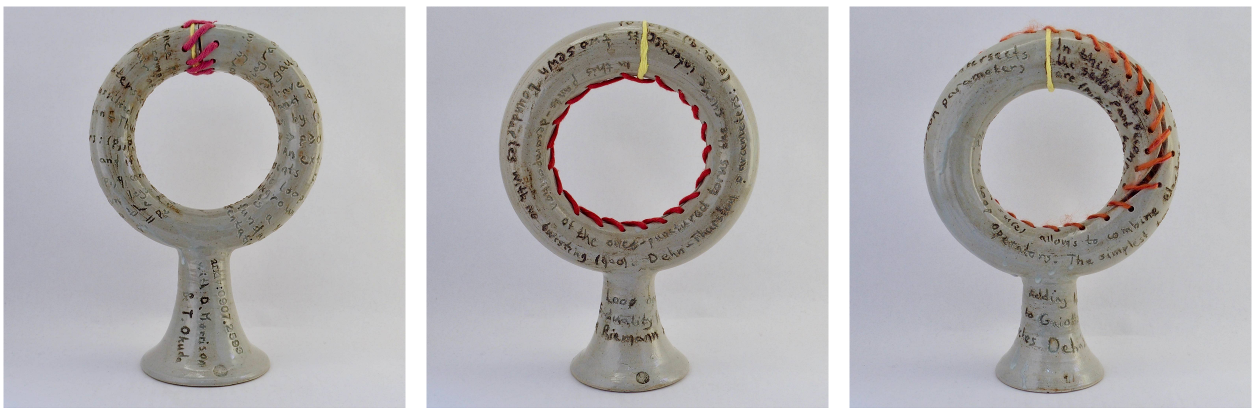


Figure 5: *"Sewing-1", wheel thrown and altered stoneware, incised, iron oxide wash, celadon glaze and yarn. Triptych. Each* $27 \times 20 \times 10$*cm.*

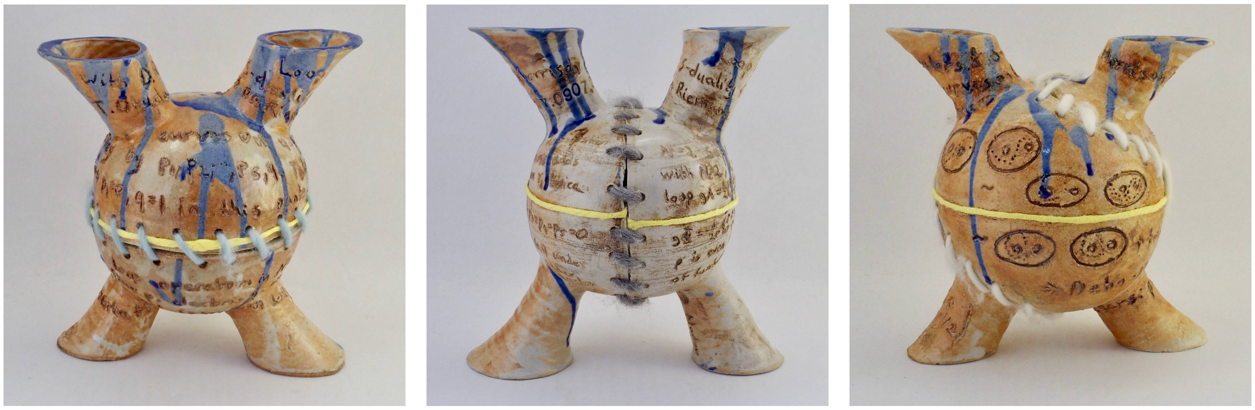
The discussion so far concerned only the pants decomposition, not the curve on the surface. In all the works above I chose the curve to wrap the small cycle of the torus close to the top, represented by the yellow strings in Figure 5. As the classification outlined above assigns numbers with respect to a particular pants decomposition, the three different pieces have different Dehn-Thurston numbers describing the same curve. Those are $(p, q) = (0, 1)$ for the image on the left $(1, 0)$ in the middle and $(1, -1)$ on the right. The details of how to read the numbers are in references [3, 1, 10, 8].

The inscriptions on these and my other works outline the calculation of these parameters for the appropriate decomposition. It also includes details of the physics version of these parameters. As discussed towards the end of this manuscript, these are line operators for particles carrying electric and magnetic charges.

Four-Punctured Sphere

Our basic building block is the pair of pants, or the three-punctured sphere. Sewing two together eliminates two of the six punctures, resulting in the four-punctured sphere. It is easy to make a spherical object on the potter's wheel, and I attach to them four tubes (similar to the single leg above) to represent the punctures. The pieces are designed to stand on a pair of those tubes with the other pair pointing upwards, see Figure 6.

As in the previous example, there is an infinite number of ways to decompose the four-punctured sphere into pairs of pants. Clay versions of three of them are shown in Figure 6. The three cuttings shown combine each pair of holes: The two bottom/top, the two right/left and the diagonals. Other cuttings connect the pairs along more complicated paths.



Horizontal cut. *Vertical cut.* *Diagonal cut.*

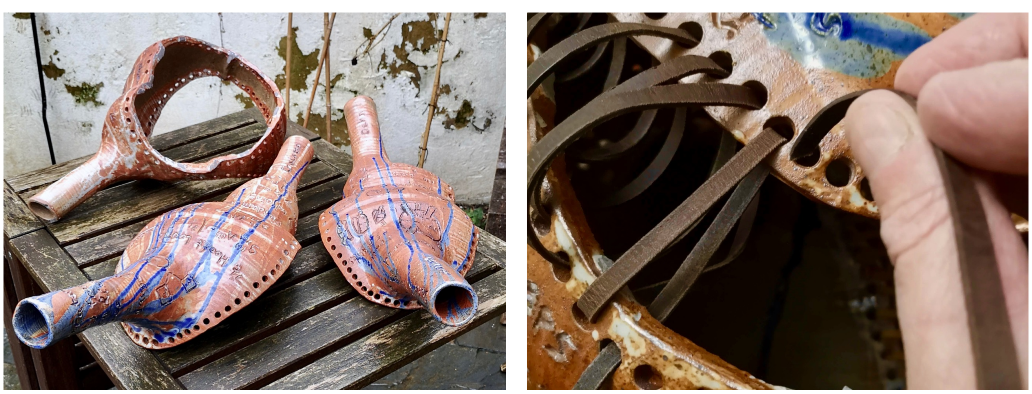
Figure 6: *Three pieces comprising the triptych "Sewing-2". Thrown and assembled stoneware, incised, iron oxide wash, shino and glue glaze and string. Each* $20 \times 18 \times 13$*cm.*

The curve on the surface is represented again by the yellow string and as before, I chose the same curve on all three pieces, but the Dehn-Thurston parameters change, as they depend on the cutting. In the cutting on the left, parallel to the string, $p = 0$ and $q = 1$. In the middle cutting, the curve intersects the string at two points, and doesn't twist, so $p = 2$ and $q = 0$. In the right-most cutting, again it intersects at two points, so $p = 2$, but the twist implies $q = 1$.

In making these pieces I had to deal with their stability and the fact that ceramics get deformed during the drying and firing stages. I started by cutting the pieces while the clay was still wet and firing them as they should stand. I put heat resistant nichrome wire hooks to hold them in place and the two pieces on the right of Figure 6 survived the firing, but one similar to that on the left and a further one deformed and were unusable. So the left one was fired as two separate pieces fired on other plates of clay that shrunk together with the hemispheres preserving their form.

Five-Punctured Sphere

A five-punctured sphere is divided into three pairs of pants along two different cuts. I have made two such pieces, one shown in Figure 1 and one in Figure 8.



The three pairs of pants after being cut. *Sewing the pieces together with leather straps.*

Figure 7: *Some steps in making the five-punctured sphere "Sewing-6".*

For stability of these pieces, I didn't fully cut them before the firing. Instead I prepared a perforated cut, leaving 1cm bridges every 5cm. The sewing holes were also pierced before the firing.

After the glaze firing, I used a diamond disc cutter to remove the bridges and separate the piece into the three pairs of pants. The results are shown at the left panel of Figure 7. I then reassembled them with leather straps, which is shown on the right panel. At the bottom left of that image, one can see on the inside of the seam an unglazed spot, which in fact is where one of the bridges was previously.

The resulting piece is shown in Figure 8. The non-selfintersecting curve, represented by the black leather straps is now composed of two disconnected pieces. One of its segments crosses both cuts and the other crosses only one. The Dehn-Thurston parameters are then $(p_1, q_1) = (4, 0)$ for the right cut in the picture, crossed four times (only two seen) and $(p_2, q_2) = (2, 0)$ for the left cut (only one seen).

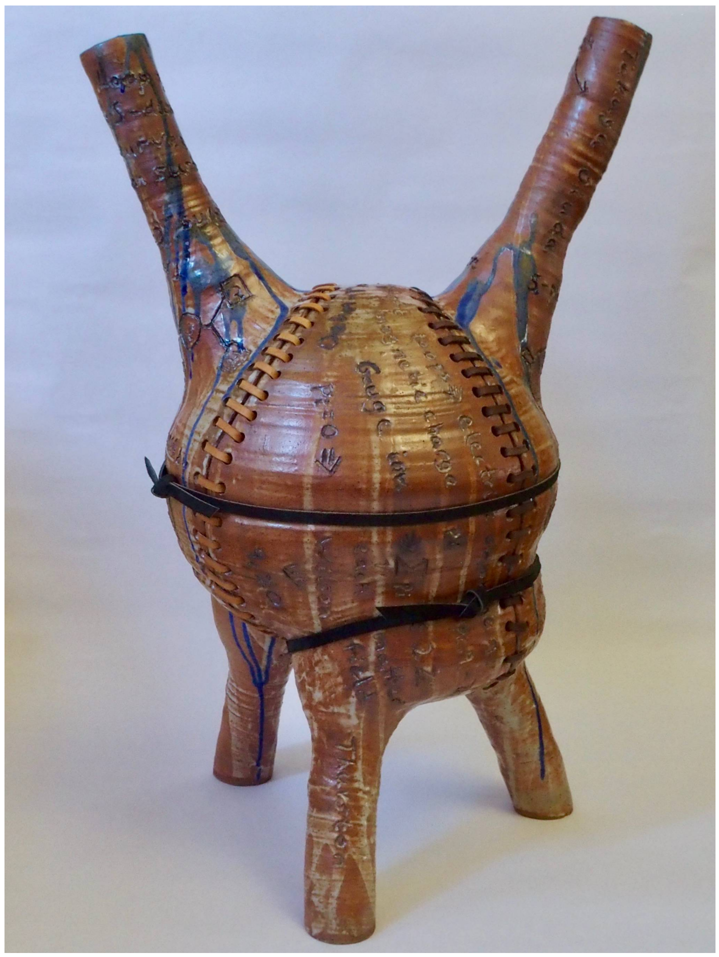


Figure 8: *"Sewing-6". Thrown and assembled stoneware with iron oxide wash, shino and blue glazing and leather straps,* $63 \times 42 \times 35$*cm. The 5-punctured sphere is assembled from three "pairs of pants".*

The Dehn-Thurston parameters for the analogous black straps in Figure 1 are $(p_1, q_1) = (2, 1)$ for the cut winding down and around the piece and $(p_2, q_2) = (0, 1)$ for the lower-right cut.

Line Operators and Curves on Riemann Surfaces

The real topic of my research [3] is the classification of line operators in certain supersymmetric field theories. It would go beyond the scope of this article to explain it in detail, but for those somewhat versed in physics, these theories are generalizations of electromagnetism, and the line operators match the possible types of particles in these theories.

There is a mapping due to Gaiotto [4] between these physical theories and Riemann surfaces, where alternative descriptions of the same underlying physical theory are related to different pants decompositions of the same surface. Our paper [3] extended the mapping to objects in these theories (the particles or line operators) and curves on the Riemann surfaces.

According to Gaiotto's mapping, each cut on the surface corresponds to one copy of Maxwell theory (to be precise an $su(2)$ vector multiplet). These theories may have electrically charged particles (like an electron) and magnetically charged ones (which are not observed in nature, though there are active searches for them).

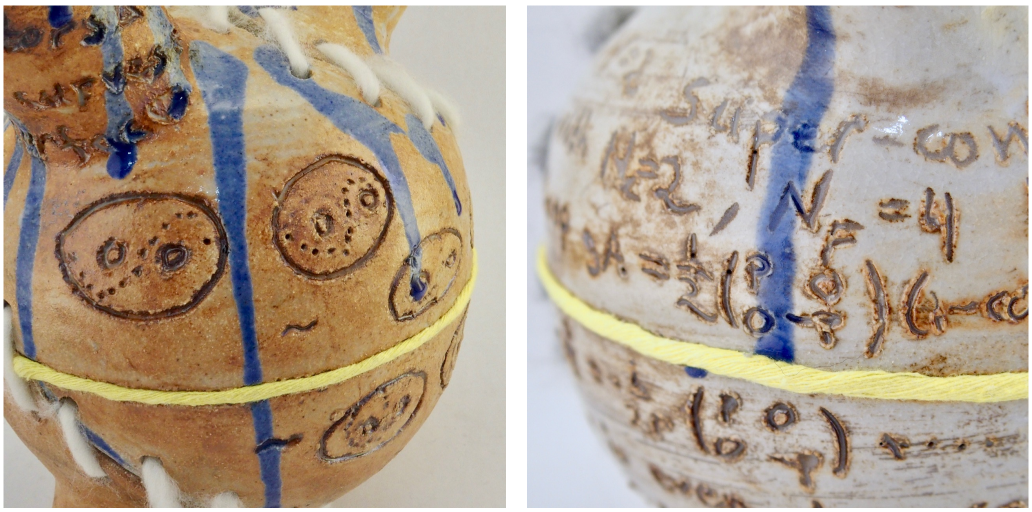


Figure 9: *Details from "Sewing-2": The ovals (left) are two pairs of pants with the dotted lines representing the yellow string. This is required for a proper calculation of the Dehn-Thurston parameters. The physics avatar is also inscribed on the pieces (right). These are values of the gauge and scalar fields corresponding to a line operator with the same quantum numbers as the curve on the surface.*

If both electrically and magnetically charged particles are allowed, so are particles carrying both charges (called dyons). There are certain conditions on the allowed charges carried by the particles (or lines), which we identified in our work. It turns out that the conditions exactly match those of the Dehn-Thurston parameters where p_i, the number of crossings, matches the magnetic charges and the twists q_i match the electric charges.

Some details of the configurations associated to these line operators are shown on the right panel of Figure 9. On the left panel are details of calculating the Dehn-Thurston parameters, where the circle with two circles inside it is a flattened pair of pants. The dotted line indicates the path of the yellow string in this particular pants decomposition.

Another question in physics is how the electric and magnetic charges change in the different descriptions of the same theory. This was first studied in [13], which supplied the answer in some cases. The full answer is given by our map to the curves on surfaces. The mapping of the Dehn-Thurston parameters for the same curve between different pants decompositions was studied in [7], thus resolving this question.

Summary and Conclusions

In this paper I presented a series of my ceramic artworks inspired by my research [3]. I tried to combine aspects of the mathematics, how it drove the design of the artworks, some details of the ceramic processes and a little bit of the theoretical physics that was the actual focus of my original research. I plan to make more artworks in this series including higher genus surfaces.

These works are part of my outreach endeavors to present advanced mathematics and theoretical physics to a wider audience. I explained more about the general philosophy at Bridges-2019 in Linz [2].

Acknowledgements

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