Appendix to paper 62 of Bridges 2021 Conference
Genesis of an Interesting Zometool-related Lattice Geometry

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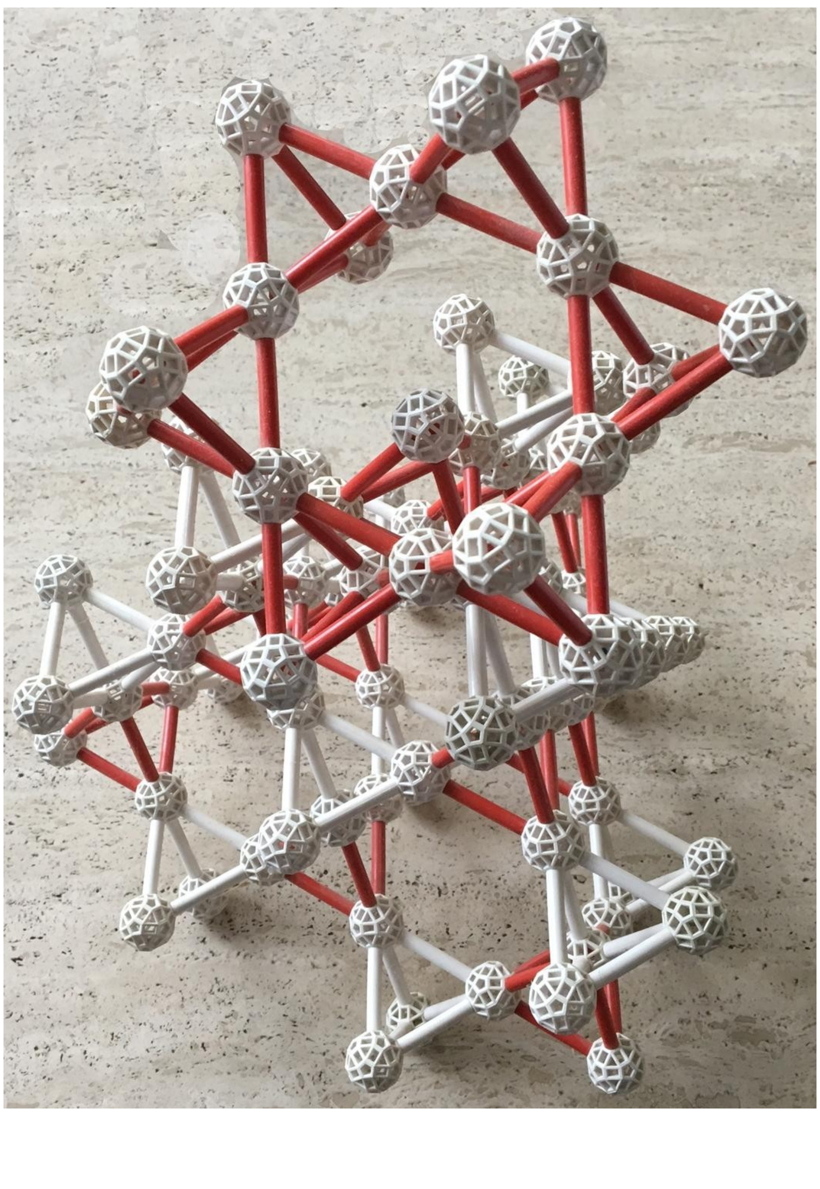
Abstract

The aim of this appendix is to get some space to elaborate with larger images on a number of points that may further interest readers closer to the subject. Also important to me as an artist and design engineer fond of geometry my personal interest of analysing the mechanisms of creativity not directly related to mathematics which are beyond my comfort zone.

Let us start with an old image of the original setup of the two models connected with a here not visible red strut broken long ago, also see further section 3 about it:



1. Why did I believe that Bogus Struts Might Open-up a Range of New Possibilities ?

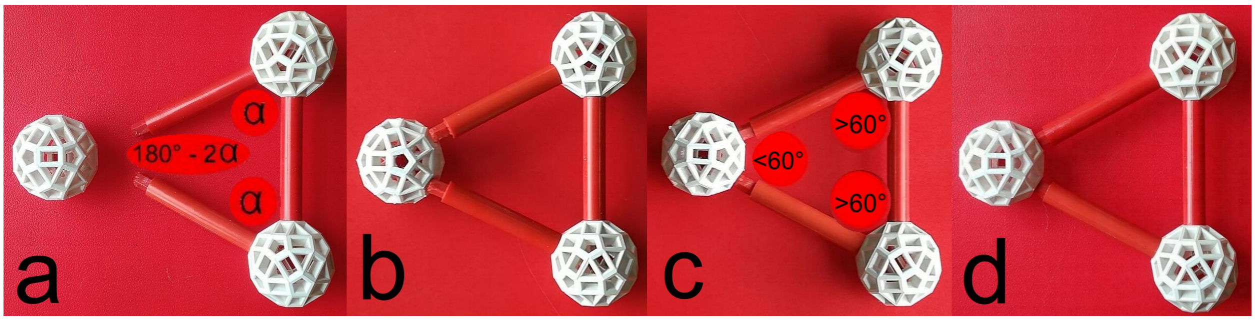
It is because:

1. Paul Hildebrandt, an immense visionary mind (which is a true statement), throws us a bunch of strange, criminal struts to playfully fool around with, in essence suggesting perhaps that he does contemplate there *must* be *something* in there ! First wrong belief: both his stature and his gesture might have made me falsely give him credit for a possible assumption that looks rightly possibly promising, but that he possibly did not have in mind at all.
2. My magic construction yet must be a *proof* for my feeling about Paul's assumption possessing some real traction (and so was still right). Second wrong belief indeed , this proof, one example, as magic as it may look, cannot guarantee a proof.
3. For years there was also Scott Vorthmann, another brilliant mind (another true statement) who brought us the vZome program that rendered possible to virtually, i.e. without the need for real Zometool inventories, to rapidly fool around and realize even more huge structures than the many ones the real version of the *Zometool system* allows. Vzome then grew boosted to tackle even more complexity beyond what this already rich system can offer ! And as Scott in the past implemented several just nice-to-have little Zometool-related desires of mine, I could now feel authorized to expect that Scott could kindly and readily accept to provide us with a new superkilling element in his vZome toolbox. It could indeed allow, despite no more real bogus struts existing, to explore this new *promising high potential* of bogus struts which I now seem to *prove* that the *something* there *thus* necessarily would be a *great one* beyond Zometool, with belief it likely directly stems from Zometool's very author Paul ! A 3rd wrong point: who am I to have believed that Scott would automatically and so easily embrace *all* my *desires*, as here vZome is proved to be deeply implemented with Zometool features.
4. Suddenly watching all those regular tetrahedra, it reminded me that the green struts just were needed for these (which is actually right), but my 4th wrong point here is coming : I never thought about precisely what the green struts offered with their new directions: sure they had the possibility of 5 new directions in the pentagonal red holes previously unable to make equilateral triangles. But so what, can't blue struts make equilateral triangles ? Yes but they can't make regular tetrahedra ! This boils down to my very wrong reality that till now I never looked at exactly *why* the green struts allowed them: as pointed out by a reviewer of my paper, *the adjacent red holes inherently sit at an orientation angle α slightly different than 60°*, with the result that such as red struts, *bogus struts can't realize 'healthy' triangles in the sense of the Zometool system law of necessarily straight struts*, a law even vZome implementation has to adhere to, hence Scott's being unable to help us with a *bogus strut functionality*.

I've been so visually attracted and delighted at the sight of such beauty and regularity concentrated in these two magical structural realizations, that my engineer and artist mind ought somehow to be blinded and fooled, up to the point of failing to see and doubtless accept a blatant impossibility yet perfectly possible because it is just standing, in pure reality, just there, in front of my sight. An unacceptable failure indeed, but *a posteriori* the humble joy of having been captured and cheated by an illusion hiding a longstanding evidence *why* this same mind perfectly knows for years that indeed *red struts can't give a triangle, hence no lattices involving triangles*, ***unless allowing unnoticeable bending***, as catched by the reviewer, and therefore his and Bridges fortunate forgiveness is indeed in place, because it is so easy to fail !

2. Kinematics & Statics of the Triangle 3rd Ball Insertion in the World of Bogus Struts

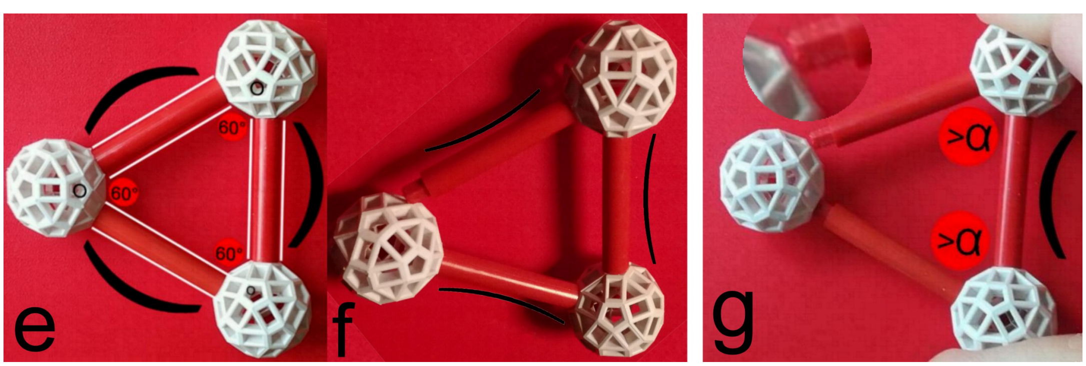
Having understood in the paper the very reason why described in the paper the final situation of a regular triangle magically made of outward bending bogus struts, it is interesting to analyse closely the kinematics of the way the loose unstressed 2 balls/3struts subsystem is going to be inserted in the 3rd ball (Figure 8a). It is magically almost an equilateral triangle with vertex angles twice α (a little larger than 60°, but magically close, despite us foolishly dreaming about having them exactly 60°) and 180° - 2α (more acute than 60°). First approaching the 2 free struts end studs to loosely touch the ball (Figure 8b) we are amazed to see the same magic acting again as we can easily rotate the ball such that it very favourably presents 2 adjacent pentagonal holes external rims just in front of the stud tips (Figure 8c) !



Figures 8a–d : *Phases of the insertion of 3rd ball.*

From there, let us consider 2 possibilities:

- In the 1st one we push the ball at once against the separated stud tips so that they start sliding in both holes (Figure 8d) while increasingly reducing the distance between studs and orienting in the α-oriented hole axes, inducing increased outward bending in all struts until completely engaged into a true equilateral triangle (Figure 8e).
- In the 2nd one, we completely engage one of the struts in any pentagonal hole of the 3rd ball which creates a non-symmetric situation where the extra length of the stud brings its tip sliding to the entry of the next hole (Figure 8f), a triangular one, with a slight force applied by a small inward bending of all struts, which increases the angle between the stud and the pentagonal hole axes. To insert the 3rd strut in its hole necessitates first taking it between fingers to extract its stud out of the interfering rim of the triangular hole (simulated in Figure 8g with a pinching and tilting at 2 1st balls), which increases the inward bending of all struts and immediately allows the fingers to bring the stud aligned with the now accessible pentagonal hole while reversing the inward bending into an outward bending that will progressively be reduced to the final one when the 3rd strut is totally engaged to again form the perfect equilateral triangle of Figure 8e.



Figures 8e–g : *Other phases of the insertion of 3rd ball.*

Of course one can have a mix of the above two ways, i.e. starting to engage the 3rd strut before the 2nd is completely inserted (Figure 8g).

There are now three pentagonal holes waiting (Figure 8e) for building a tetrahedron along the same ways, and from there on one automatically finds the holes that are going to construct the lattice because there are no other possibilities. I don't remember any more how the model was finally shaped in some nice *meta*tetrahedron. All I can recall is my amazement having magically been able to use exactly all the struts received including the last little tetrahedron which I recently found (see Figure 11 at the end of the Appendix 6th and last section), but the magic finishes there, as the final little tetrahedron is now irregular colourwise because it finally has 4 red struts and 2 whites !

3. Story of the Only Broken Bogus Strut in the Original Model

It might be interesting to dig into an apparently strange situation: remembering that my small model was originally linked to the large one by my last bogus strut, this strut was *the only one to break*. As for all the other struts they seemed to perfectly adhere to the *straight struts* orthodox Zometool law. So to decide it broke because more stressed than the others is quite complex to verify, as it depends on many parameters including material properties. In any case, the bending probably also affects the balls themselves. In reality here, the ball centers look mutually 'forced' by 3D symmetry to remain in a regular tetrahedral symmetry, into the four different orientation families mentioned earlier by observation.

4. Another Way to Explain to Deep Zometool Specialists the Bogus Struts Bending Source

Let us now try to show the mechanics of how it works. Figure 9 displays part of a green struts perfect regular tetrahedral lattice (of which my two models are now a part) that Scott Vorthmann constructed with vZome using 4 regularly distributed ball colours, 4 per tetrahedron, on 6 line families of alternative color pairs [9]. Looking at one of the tetrahedra forming the lattice on the detail picture, we must first remove a green edge, here between a red ball and a black one, to free their pentagonal holes.We can now insert in their place two *short red strut studs* which evidently sit together in another plane than the faces containing both balls sitting in their green struts tetrahedral and thus equilateral triangles environment. They necessarily do not align due to above α angle slightly larger than the 60° of the triangular faces that contain the two balls. Next let us add two *yellow struts* that fit in the triangular holes holes against the red studs, that by Zometool essence are 'normal' to the tetrahedron triangular planes facing these holes [7]. They will be used as rotational axes to virtually bring with small rotations the studs and both balls, within closest tetrahedron faces. But the studs are still not aligning yet in near opposition close to the axis of the removed tetrahedral edge green strut, so showing the small gaps all the bogus struts will need to fill together symmetrically when forced to small bending. If the same rotations are provided for all such small red struts, their holes will approximately be oriented symmetrically for receiving the symmetrical bogus struts forcing announced small bending. It looks now that tetrahedral symmetry (with possible loose additional structures) is likely the only use of bogus struts [7].

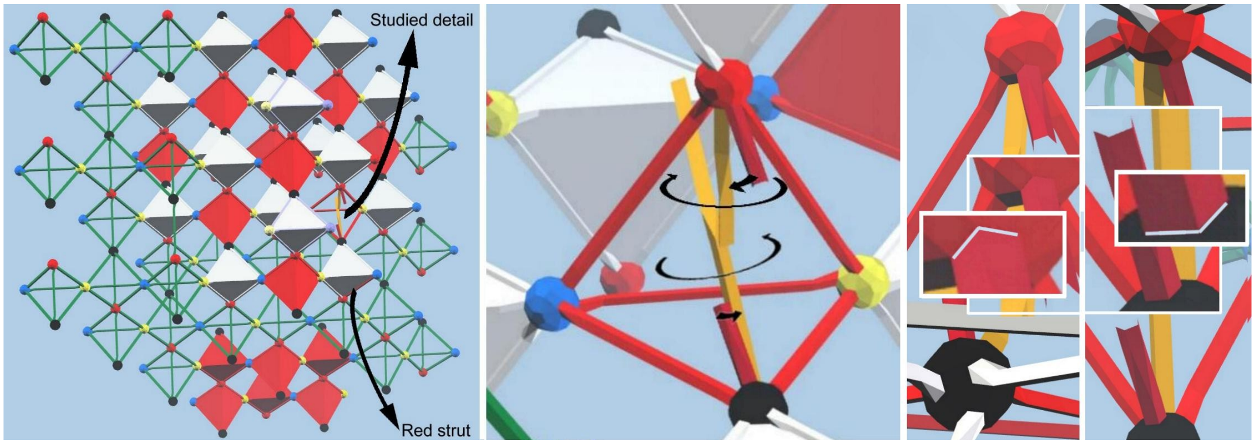


Figure 9 : *Scott' s lattice , embedded tetrahedron studied and details of red and black ball turned.*

5. Other Bogus Struts Discovered : New Magic !

I suddenly discovered a dust covered bogus tetrahedron on colored balls, that could be the very first thing I constructed with my set of bogus struts. 3 reds and 3 whites, with a little welcoming paper figure pinned on a ball, I immediately imagined it saying “Hello World !”.

Sure enough, I soon saw I could take advantage of one of the whites to replace “the wrong red that made the model not entirely symmetric” ! Unfortunately (see Figure 10), I broke 3 struts, the red I immediately placed in the newfound tetrahedron, to save a white to replace the removed red in the large model to be repaired. I will need to cut off an end stud of this white because after having glued the broken part in place, I will no longer have the possibility to deform anything if I want to save forever the then perfectly symmetric large model !

This allowed me to not mention any more in the paper the story of this red bogus strut I was obliged to insert in the large model instead of a no longer available white...

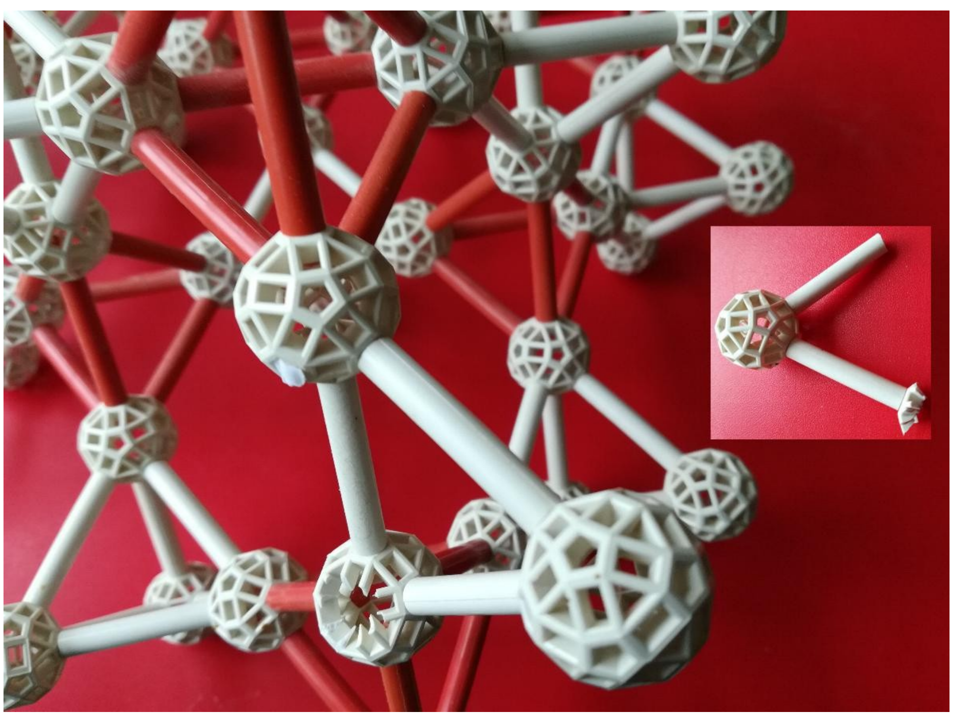


Figure 10: *Catastrophe at the large model...*

6. Almost Important Last Minute Additional Magic

I was also suddenly tempted, at the sight of this result and inspecting carefully again the small model, to try continuing and reinforcing the magic already contained in this deliberately detailed timeline. Looking at those unused pentagonal ball holes the model has, I remembered I owned quite a large inventory of *ultra-short* red struts developed a few years after the super-short ones. I could almost fill all the holes (see Figure 11) and it really started to seem possible, at the sight of mutual correspondence of all these facing studs, that if I had more bogus struts they might fill the model, thanks to bending, with a continuous *bogus tetrahedral lattice* which is impossible with the green struts. Now with these 6 dusty extra bogus struts forming a tetrahedron I could just give the idea a try, and so completed another tetrahedron above one looking down and where 3 other pentagonal holes look ready to also accept bending bogus struts and *...it works !!!* But it didn't get further, the upper vertex being too far away from the lower three vertices. So gone was the 'continuous bogus tetrahedral lattice' which is thus also as impossible with the bogus struts as it is with the greens ! This is *unfortunate* for the lost further magic but *fortunately* adheres to rigorous math saying "regular tetrahedra don't fill/pave 3D space"!

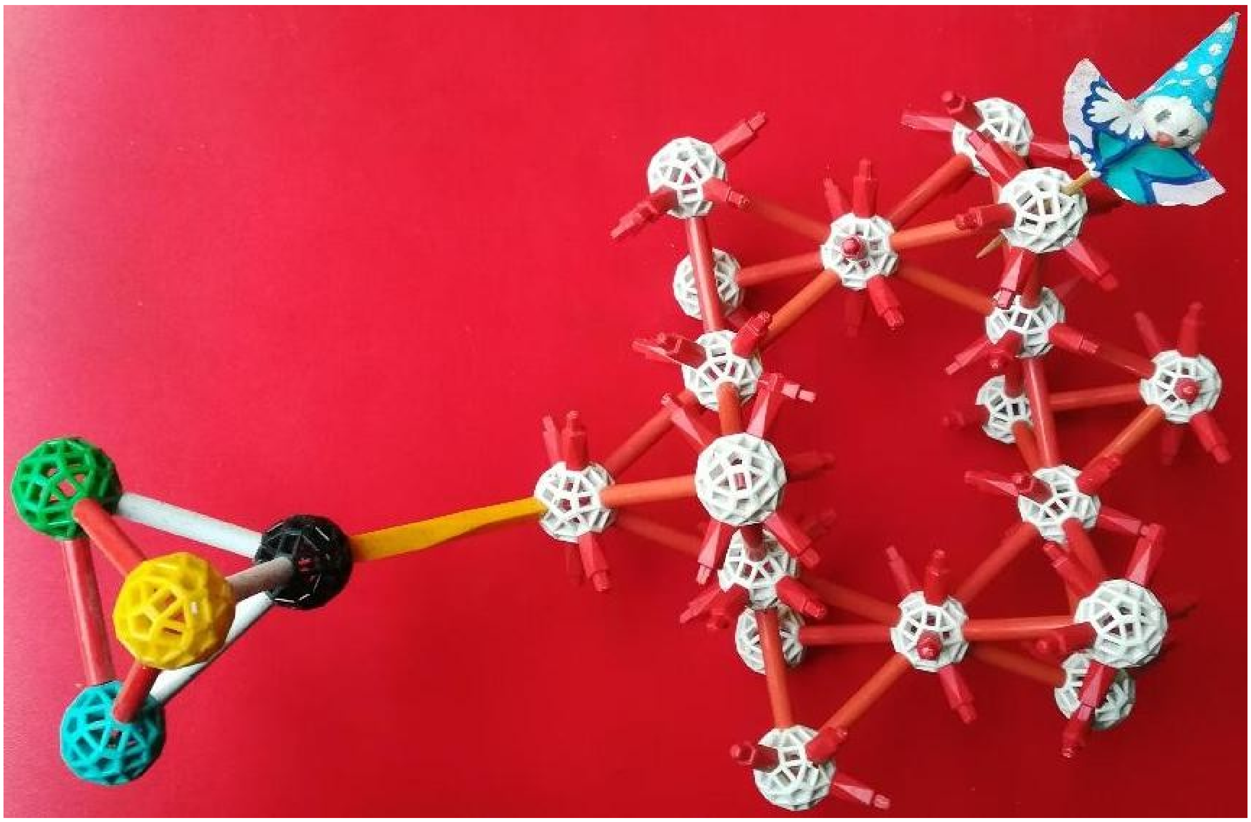


Figure 11: *Red holes filled, 6 other bogus struts found, Hello World !*

Incidentally I now remember that a long time ago, playing with *Polydron* equilateral triangles, I was astonished that I couldn't close a loop of 5 contiguous tetrahedra: there exists a gap. I later observed when building an icosahedron with blue struts, starting from a vertex I could only reach the center using red struts (which is actually a way to teach kids building icosahedra starting from a ball filled with reds, see [1] and Zometool tutorials). So the inside triangles of the icosahedron are made by 2 adjacent red struts closed by a blue strut, and are not equilateral. But let's look closer at what happens here. In our large model there are 4 truncated tetrahedral empty volumes, between the 4 white bogus tetrahedra. Of the

4 inside white empty triangles, only 1 can become a tetrahedron, as its ball inside is “not central” to the truncated space, which confirms already the *non-existence of a dense tetrahedral lattice*, but not distant enough from the center because it would interfere with the inside balls of the 3 other candidate tetrahedra and so eliminates also the possibility of 4 balls “kissing” in the center.

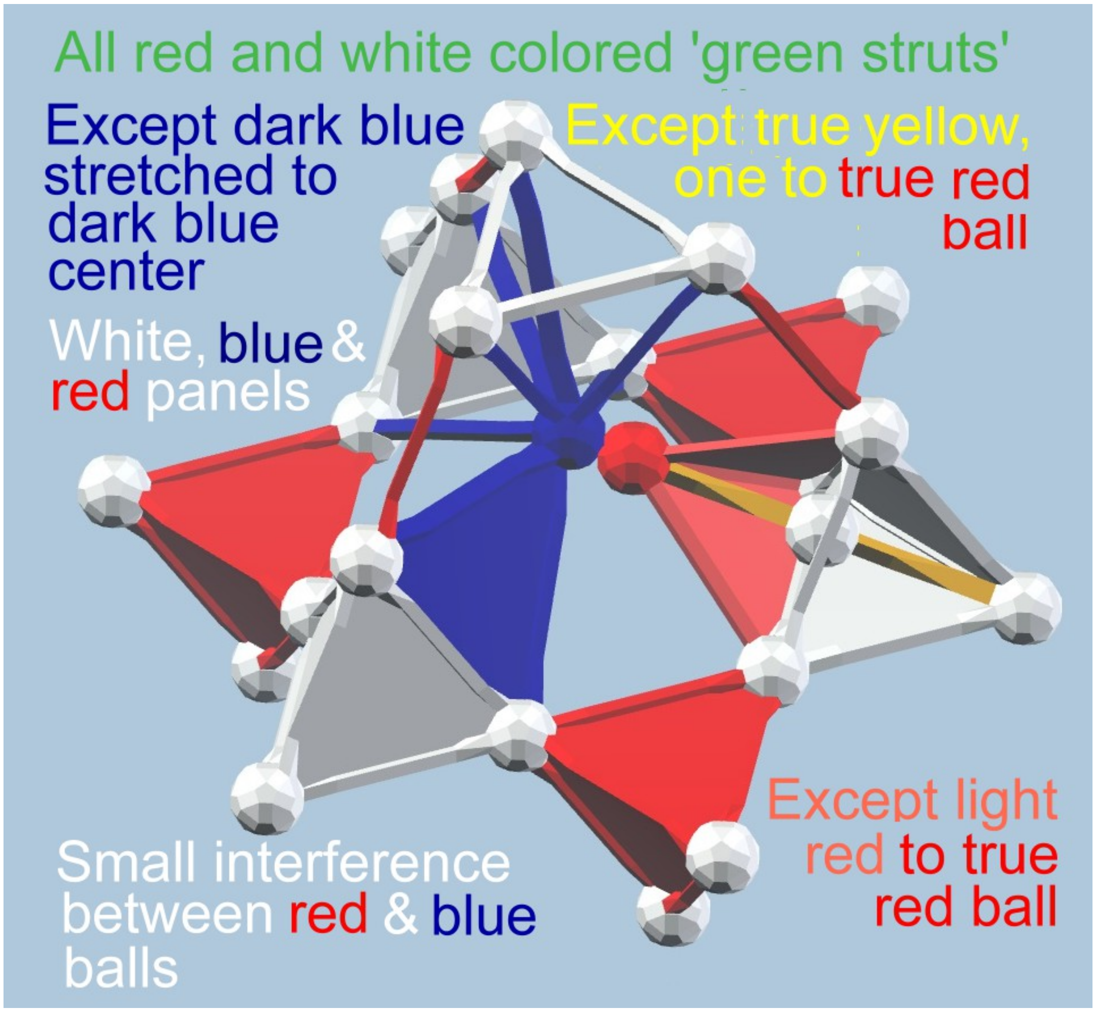


Figure 12: *vZome image with explanations for impossibilities.*

To show this I made a vZome model with colored regular green Zometool struts of an under part of our large model showing the truncated tetrahedron empty space where I sought the normal to the internal face of a white tetrahedron and found of course a regular yellow strut but where its end ball is the symmetry of the outside vertex of the white tetrahedron, so the ball is exactly the one of the inside tetrahedron we want to construct, which can easily be done by vZome as 3 non-regular struts each being built between two existent balls (see Figure 12). Strut bending works in our existing lattice, because allowing struts to adjust their orientation themselves 3-dimensionally by bulging outside the elementary tetrahedra, so shortening the distance between balls. If the 3 bogus struts of the extra tetrahedra were much more elastic they could stretch by traction to reach the center of the open volume quite far away (the gaps problem) without perturbing the equilibrium reached by symmetric bending. Said *center, not a member of a regular lattice* is obtained in the vZome model of Figure 12 as the centroid of the 4 balls now hidden but themselves the centroids between opposite outer vertices of the white tetrahedra around the void plenum. I certainly now will no longer tend to ask Paul Hildebrandt to please bake me some more bogus reds if the present tooling still can easily be used in the same erroneous way...

7. **Enlarged Versions of the Images of the Paper**

Figure 1: *Struts Compared .*

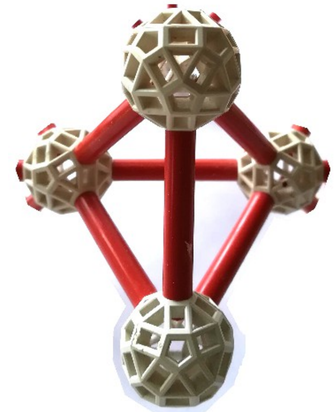


Figure 2: *Ball orientations.*

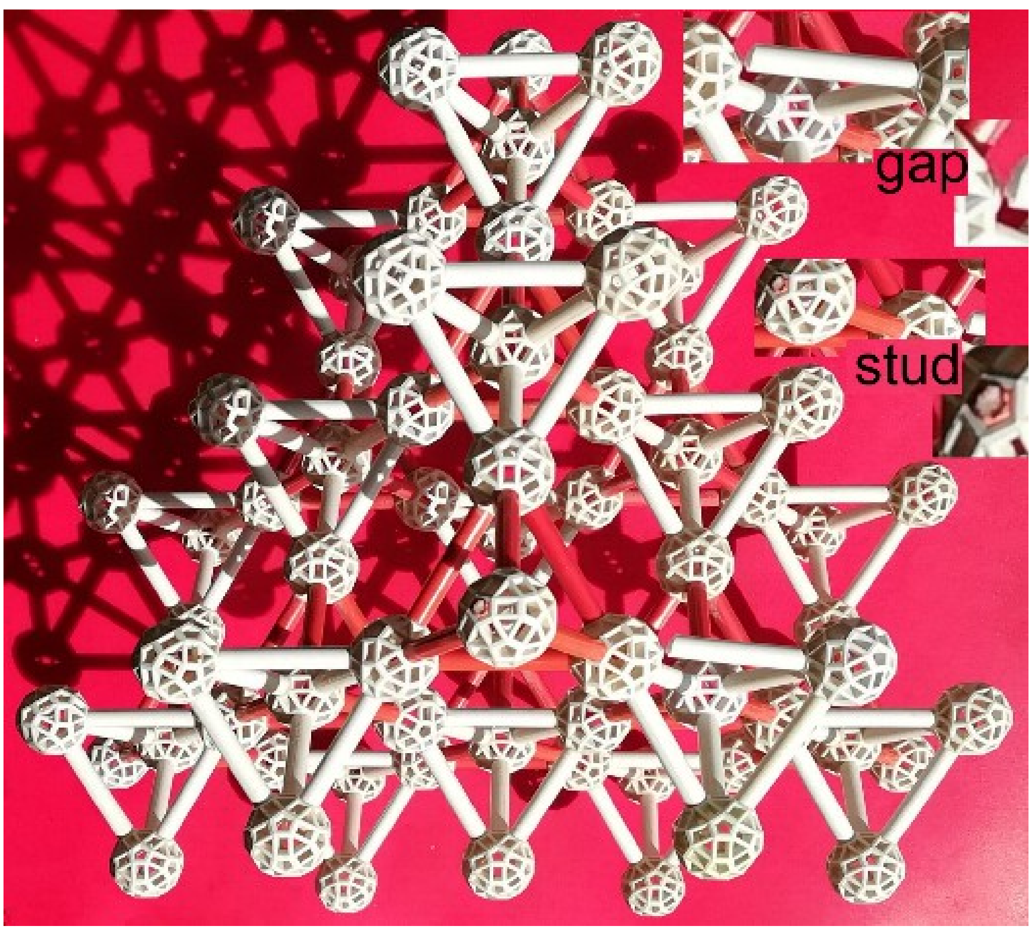


Figure 3: *large bogus model and details as "gap" due to broken white strut and remaining "stud" of red strut that connected 2 models, broken long ago.*

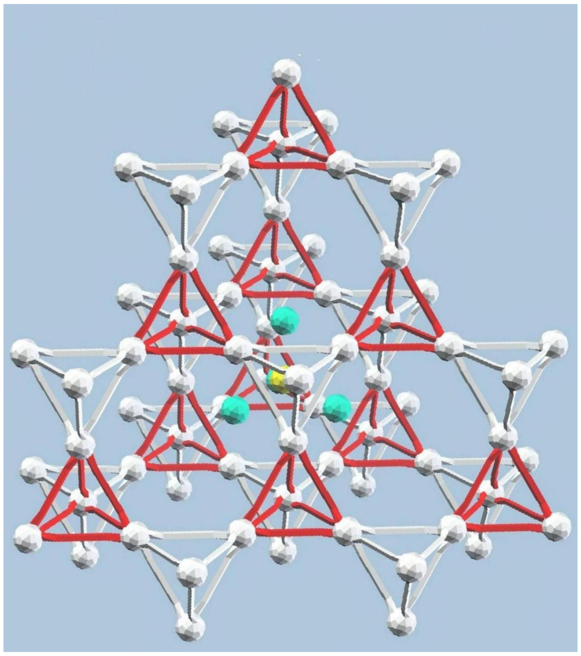


Figure 4: *large vZome model.*

Figure 5: *large Zometool model.*

Figure 6: *Different views of red & bogus struts before getting triangulized and Large red struts when +/- triangles.*

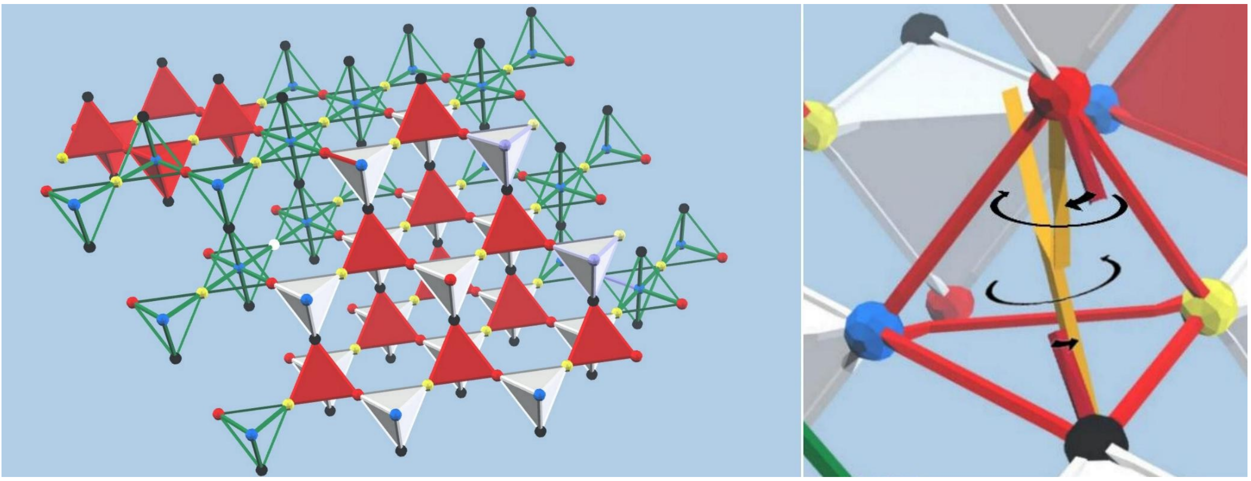


Figure 7: *Paneled models built in green lattice showing four families of balls,*
and detail of all balls to slightly turn along yellow axes to get best oriented for receiving the bogus struts slightly oriented for receiving the bent bogus struts.

Additional reading

[9] https://www.pbs.org/wgbh/nova/transcripts/2414proof.html mathematicians can get emotional !

[10] https://www.youtube.com/watch?v=qiNcEguuFSA Andrew Wiles emotional I short clip.

[11] https://www.dailymotion.com/video/x1btavd Andrew Wiles' Fermat adventure in long movie.

[12] M. Senechal. “Which Tetrahedra Fill Space?” *Mathematics Magazine*, vol. 54, no. 5, 1981, pp. 227–243. *JSTOR*, www.jstor.org/stable/2689983 . Accessed Apr. 29, 2021, to show some epistemology.