Approximating Logarithmic Spirals by Quarter Circles

Ulrich Reitebuch[1a], Martin Skrodzki[2b], and Konrad Polthier[1c]

[1]Department of Mathematics and Computer Science, Freie Universität Berlin, Germany
[a]ulrich.reitebuch@fu-berlin.de, [c]konrad.polthier@fu-berlin.de
[2]Computer Graphics and Visualization, TU Delft, The Netherlands
[b]mail@ms-math-computer.science

Abstract

The approximation of a golden logarithmic spiral by quarter circles is well known. Starting from this, we show that any logarithmic spiral can be approximated by quarter circles in a similar way. Using our construction on a rectangle with aspect ratio $\sqrt{\phi}$ and performing a coordinate reparametrization, we obtain an aesthetic partition of the plane as our main artwork.

Logarithmic Spiral

Logarithmic spirals, or approximations of them, appear in various more or less natural structures. Examples include the shell of a nautilus, the center of a cyclone, the (spiral) arms of galaxies, phyllotaxis patterns of plants, or substructures in the Mandelbrot set, see Figure 1. Constructions with logarithmic spirals have been (re-)visited in several previous *Bridges* publications. To name some examples, in [7], the logarithmic spiral is the underlying inspiration for a virtual *Spiral Surface* model. The use of logarithmic spirals in architecture is considered in [2]. Somewhat similar to this paper, [5] extends a classical construction of the logarithmic spiral with triangles to a construction with general *n*-gons.


Figure 1: *Examples of more or less natural approximations of logarithmic spirals. From left to right: nautilus shell, center of a cyclone, spiral arm of a galaxy, phyllotaxis pattern in romanesco broccoli, and a part of the Mandelbrot set (which approximates a logarithmic spiral as discussed in [6]). All images taken from pixabay.com.*

The logarithmic spiral is—from a mathematical viewpoint—a special curve, which grows out from its center by a constant factor per angle. It can be parameterized by $\gamma(t)$, with derivative $\dot{\gamma}(t)$, as follows:

$$\gamma(t) = \begin{pmatrix} \cos(t)e^{at} \\ \sin(t)e^{at} \end{pmatrix} \qquad \dot{\gamma}(t) = e^{at}\begin{pmatrix} a\cos(t) - \sin(t) \\ a\sin(t) + \cos(t) \end{pmatrix} \tag{1}$$

with some constant $a \in \mathbb{R}$. Here, $a = 0$ corresponds to the degenerative case in which the spiral collapses into a circle. For $a < 0$ the curve spirals inwards, converging at the origin, while for $a > 0$ the curve spirals
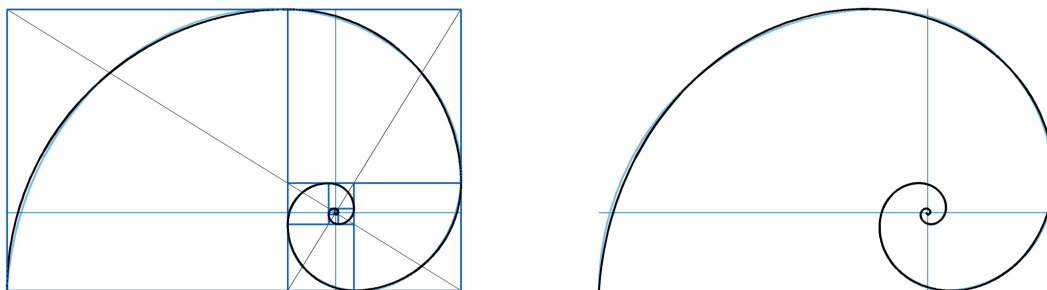
(a) *Approximation of the golden spiral (blue, background) by quarter circles (black, foreground).*

(b) *An actual golden spiral (black, foreground) and its approximation by quarter circles (blue, background).*

Figure 2: *Comparison of the golden spiral and its approximation via quarter circles. The difference is most notable around the first and largest quarter circle.*

outwards for growing t. The angle between $\gamma(t)$ and $\dot{\gamma}(t)$ is constant for all t, as the following calculation shows, where $\langle v, w \rangle$ denotes the inner product of two vectors v and w:

$$\angle(\gamma(t), \dot{\gamma}(t)) = \cos^{-1}\left(\frac{\langle \gamma(t), \dot{\gamma}(t) \rangle}{\|\gamma(t)\| \, \|\dot{\gamma}(t)\|}\right) = \cos^{-1}\left(\frac{ae^{2at}}{(e^{at})\left(e^{at}\sqrt{a^2+1}\right)}\right) = \cos^{-1}\left(\frac{a}{\sqrt{a^2+1}}\right).$$
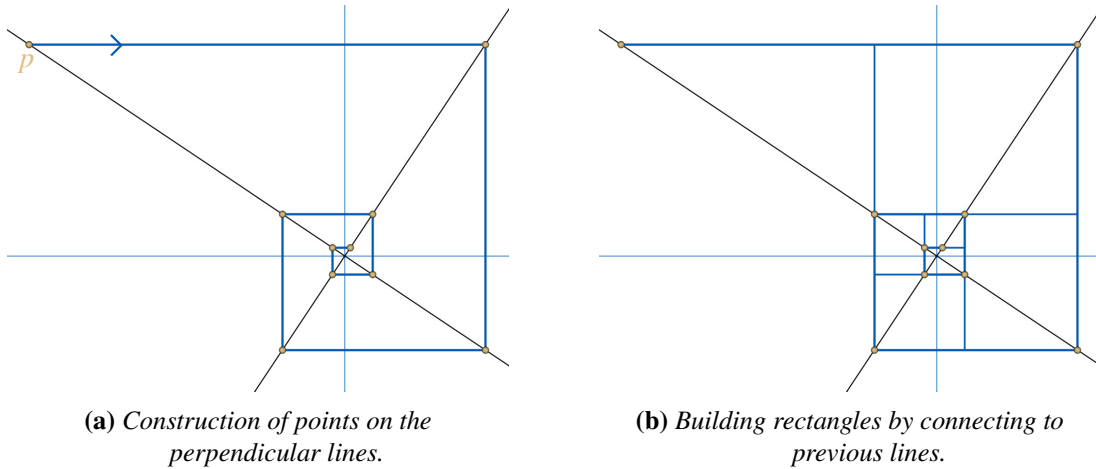
The logarithmic spiral with parameter $a = 2\ln(\phi)/\pi$ scales by the golden ratio $\phi = (\sqrt{5}+1)/2$ for a quarter turn in counterclockwise direction and is often called the *golden spiral*. While the nautilus shell as shown in Figure 1 is frequently cited as an example of a golden spiral in nature, the evidence for this claim is mostly anecdotal [3]. In fact, studies have found several ratios for different individuals as well as for varying species of nautilus, see [1]. Similarly, the other examples from Figure 1 are mere approximations of the golden or respective other logarithmic spiral.

In the remainder of this paper, we start by recalling the classical approximation of the golden spiral via quarter circles. A generalization of this construction allows us to approximate arbitrary logarithmic spirals. These can in turn undergo a reparametrization which creates a tiling of the plane and our final artwork.

Quarter Circles for Approximating a Golden Logarithmic Spiral

For the classical construction, we start with a golden rectangle, cut off a square, and inscribe a quarter circle. The rest of the initial golden rectangle is again a golden rectangle, rotated by $-\pi/2$ and scaled by $1/\phi$. The construction is iteratively repeated with this smaller golden rectangle, see Figure 2a. This approximation yields a curve fairly close to an actual golden spiral as shown in Figure 2b.

Every two adjacent quarter circles have the same tangent direction at their common end point. The collection of all quarter circles gives an approximation for a golden spiral, see Figure 2. The limit point for the golden spiral is the intersection point of diagonals of the first and second golden rectangle. Note that these diagonals are the same for all following smaller golden rectangles. When the initial golden rectangle is drawn parallel to the coordinate axes, the slopes of the diagonals are $-1/\phi$ and ϕ, so the drawn diagonals of the first and second golden rectangle are perpendicular to each other.

(a) *Construction of points on the perpendicular lines.*

(b) *Building rectangles by connecting to previous lines.*

Figure 3: *Construction from two perpendicular lines with slopes $-\frac{2}{3}$ and $\frac{3}{2}$.*

Construction from Diagonals

We have seen how a series of nested golden rectangles gives rise to their diagonals and thus a common intersection point. However, the construction can also be reversed. That is to say, the sequence of rectangles can also be constructed from the two diagonal directions.

The initial setup consists of two perpendicular lines through the origin—one with slope $-s$ and one with slope $1/s$ for some $0 < s < 1$—and an arbitrary point $p \neq 0$ on the line with slope $-s$. Starting from this point, we draw a line segment parallel to the x-axis towards the other line until intersection, which we mark with a new point. This new intersection point is in turn the start of a new, line segment parallel to the y-axis, see Figure 3a for an illustration of this process with two lines of slopes $-\frac{2}{3}$ and $\frac{3}{2}$.

By iterating this process, we get an infinite sequence of points, alternating on the two lines. In each step, the figure is similar to the previous step by rotating by $-\pi/2$ and scaling by s. Starting from the fourth iteration step, i.e., the fourth line segment, we extend the line segment beyond the intersection point. These line segments stop on the first encountered, previously drawn line segment. Thereby, the construction yields a set of nested rectangles, see Figure 3b. However, these rectangles are generally not squares and thus, we cannot simply inscribe a quarter circle into each.
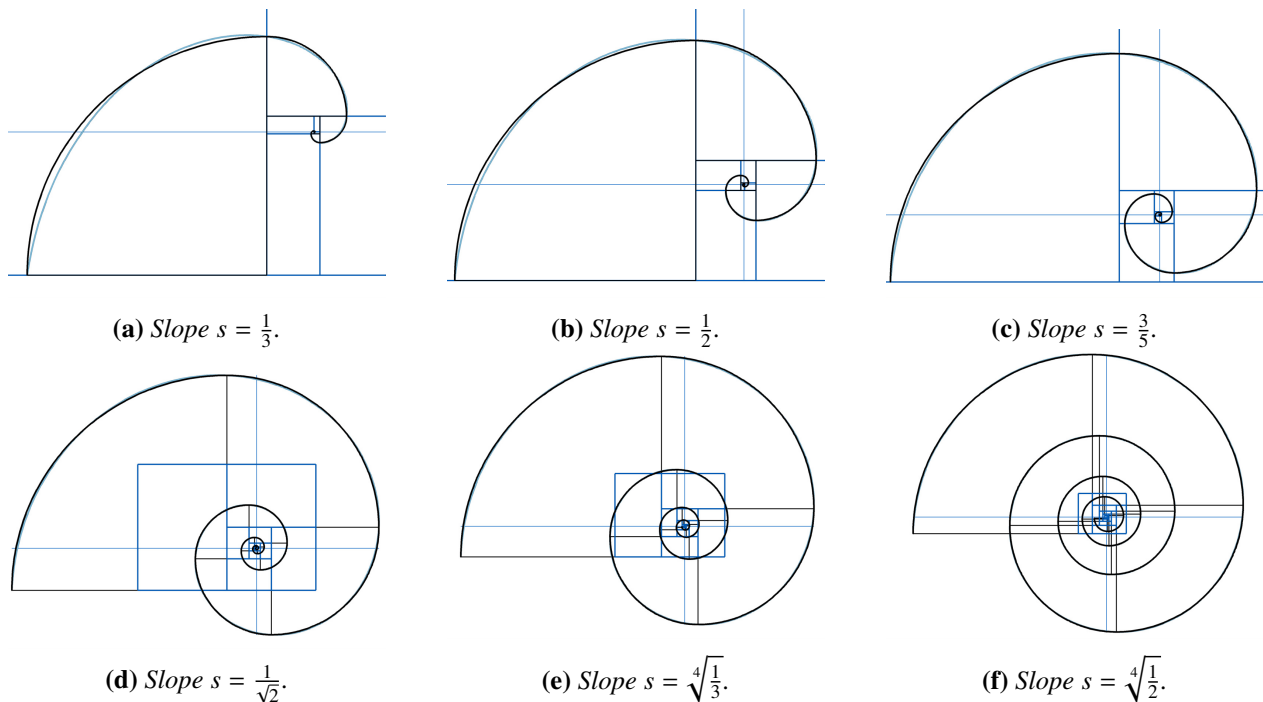

(a) *Constructing the radius.*

(b) *Approximation by quarter circles.*
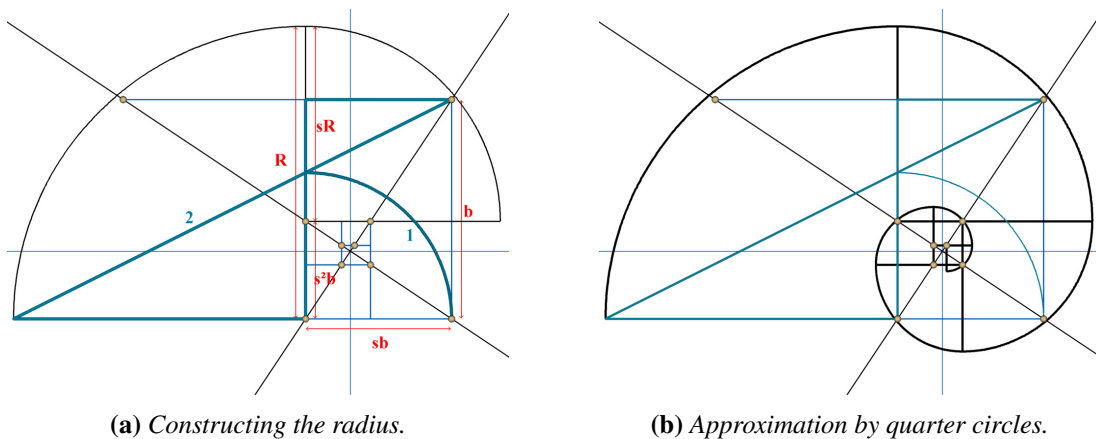
Figure 4: *Constructing the approximation by quarter circles, starting from Figure 3b.*

We want to construct the radius of a quarter circle such that the next quarter circle has the same tangent at their common intersection point and is scaled by factor s. Therefore, the radius R has to be the same as the next radius plus the height of the remaining rectangle of the next step. When the current rectangle has height b, we get:

$$R = sR + s^2 b \qquad \Leftrightarrow \qquad R = \frac{s^2 b}{1 - s}.$$

Geometrically, this radius R can be constructed by the intercept theorem via the following steps. First, construct a point on the vertical edge cutting of the first rectangle, at height sb over the lower edge. This can be done by drawing a quarter circle of radius sb around the lower intersection point on said vertical edge, see marked **1** in Figure 4a. Second, construct the intersection of the line through this point and the upper right corner (see marked **2** in Figure 4a) with the straight line extending the lower horizontal edge. A quarter circle of radius R can then be drawn around the lower intersection points on the vertical edge by setting the compass to the intersection point constructed in the second step.

Given this procedure, we can construct an approximation by quarter circles to any logarithmic spiral with $a > 0$. Note that by going to the mirror image of our construction, we can also approximate logarithmic spirals with $a < 0$. From the computation of R it is clear that every quarter turn in clockwise direction scales the approximation by a factor of s. In terms of the constant a as used in the parametrization of a logarithmic spiral as given in Equation 1, it is therefore $a = -2\ln(s)/\pi$. In other words, to approximate a logarithmic spiral with a given constant a, we have to choose $s = e^{-a\pi/2}$. As for the starting point, we can pick any point as the initial point of the construction and perform the procedure to obtain an approximation of the spiral (which already has the correct s by choosing the corresponding diagonal). If it is desired to approximate a

(a) *Slope $s = \frac{1}{3}$.*   **(b)** *Slope $s = \frac{1}{2}$.*   **(c)** *Slope $s = \frac{3}{5}$.*

(d) *Slope $s = \frac{1}{\sqrt{2}}$.*   **(e)** *Slope $s = \sqrt[4]{\frac{1}{3}}$.*   **(f)** *Slope $s = \sqrt[4]{\frac{1}{2}}$.*

Figure 5: *Approximations of different logarithmic spirals following our construction for varying slope values s. Note that the spirals in Figures 5e and 5f scale by one third and one half on a full clockwise turn respectively. The corresponding actual logarithmic spiral for each slope is given lightly colored in the background, most notably for the smaller slopes in Figures 5a and 5b.*
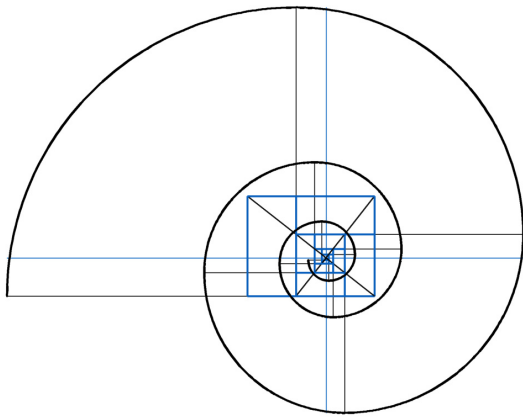
(a) *Approximation of a "better golden logarithmic spiral" by quarter circles.*
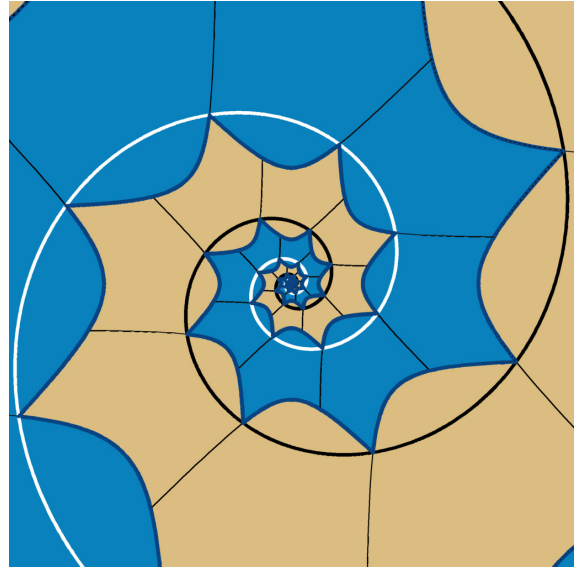
(b) *Reparametrization of Figure 6a to an approximated golden spiral.*

Figure 6: *An approximation of a logarithmic spiral, build from "better golden rectangles" and its reparametrization to an approximate golden spiral. The right image also includes a copy of the reparametrization, rotated by π.*

specific logarithmic spiral (not only one with the correct slope), the obtained approximation might need to be rotated—or scaled, equivalently—to reach the desired spiral.
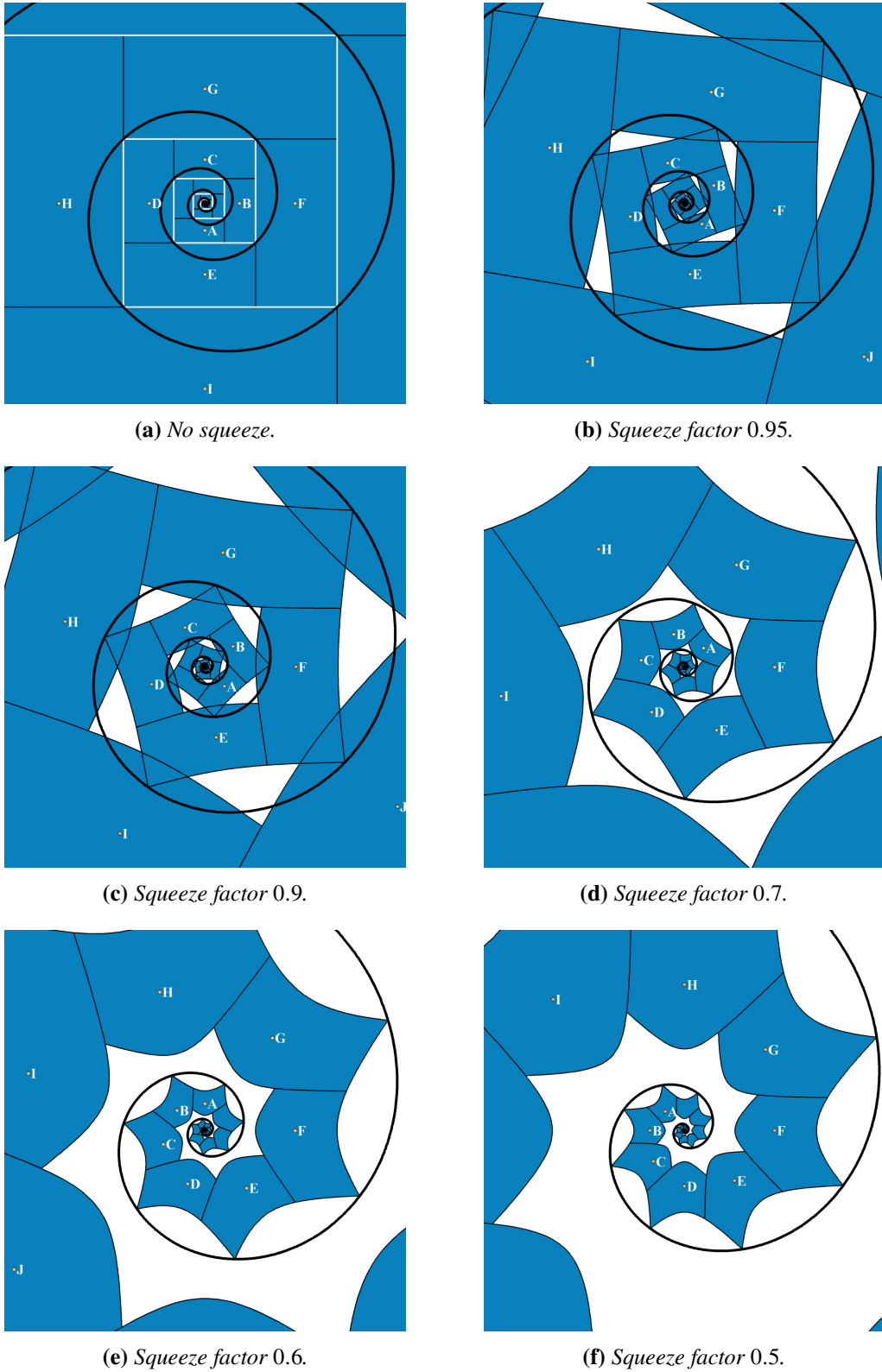
Note that this construction is a generalization of the classical approximation of the golden spiral. That is, for $s = 1/\phi$, it yields the well known approximation. Similar to the procedure presented in [4], our construction is also capable of producing myriad different approximations, see Figure 5 for a selection.

"Better Golden Spiral"

In his 2018 Bridges publication, Douglas McKenna proposed to use a *"better golden rectangle"*, with aspect ratio $\sqrt{\phi}$, see [4]. Figure 6a shows our approximation of a *"better golden spiral"*, which is constructed from such "better golden rectangles", scaling by ϕ^2 per winding in counterclockwise direction.

This construction yields a curious connection between the approximation of the "better golden spiral" and that of the golden spiral. When we transform the whole construction in Figure 6a to polar coordinates, then squeeze the angle parameter by a factor $\frac{1}{2}$, and transform back to Cartesian coordinates, the resulting construction covers only half of the Euclidean plane, see Figure 7f.

Note that this reparametrization has to be performed carefully in order to keep the spiral intact. That is to say, using a 'simple' representation in polar coordinates and considering half of the angular coordinate, the entire plane is mapped to the right half-plane: For instance, cut along the negative part of the x-axis, i.e., providing the angle as $[-\pi, \pi)$ with respect to the positive part of the x-axis (angle 0 keeping the x-axis fixed and positive or negative angle covering the upper or lower half-plane respectively). Considering half of the angular coordinate, the entire plane would be mapped to the right half-plane (with positive x-coordinates). Thereby, the spiral as well as the constructed rectangles and their approximating quarter circles would be cut by the negative part of the x-axis.

(a) *No squeeze.*

(b) *Squeeze factor* 0.95.

(c) *Squeeze factor* 0.9.

(d) *Squeeze factor* 0.7.

(e) *Squeeze factor* 0.6.

(f) *Squeeze factor* 0.5.

Figure 7: *Snapshots of the reparametrization of an approximate "better golden spiral" to an approximation of the golden spiral. The point F is a fixpoint of the reparametrization. ̈In the last image, the angle parameter is squeezed by a factor of $\frac{1}{2}$ and thus all angles are halved with respect to F.*

Therefore, in order to keep the spiral intact, we cut the image not simply along one part of the coordinate axes, but along the sequence of successive line segments as constructed in Figure 3a and as shown in white in Figure 7a. To illustrate the effect of this reparametrization, we have labeled a set of points $A, \ldots, J$ which are in the same relative position within their parent rectangles of Figure 7, where the point F will be a fixpoint of our reparametrization. As we cut along the white line segments of Figure 7a, in our initial construction, the labeled points have the following angles with respect to the origin in the center and the fixpoint F, which sits at angle 0: (E: $-\pi/2$; D: $-\pi$; C: $-3\pi/2$; B: -2π; A: $-5\pi/2$). Note that, unlike in regular polar coordinates, these values can be smaller than $-\pi$ or larger than π. This is, because we take the windings of the spiral into account. Squeezing these coordinates by a given factor shortens their relative angles by the chosen factor. The corresponding effect of this reparametrization is shown in the snapshots given in Figures 7b to 7f. By squeezing the angle parameter, the initial sequence of white line segments gradually opens to eventually become a connected, white space. Finally, in Figure 7f, all angles have been halved: (E: $-\pi/4$; D: $-\pi/2$; C: $-3\pi/4$; B: $-\pi$; A: $-5\pi/4$), e.g., point B has now moved from being on the same line from the origin as the fixpoint F to the opposite site of the origin, onto the negative part of the x-axis.

Note how the initial construction in Figure 7a tiles the entire plane, but as we squeeze by a factor of $\frac{1}{2}$, the reparametrized object covers only half of the plane, see Figure 7f. In addition to keeping the spiral intact, by performing this reparametrization the approximate "better golden spiral" is mapped to an approximation
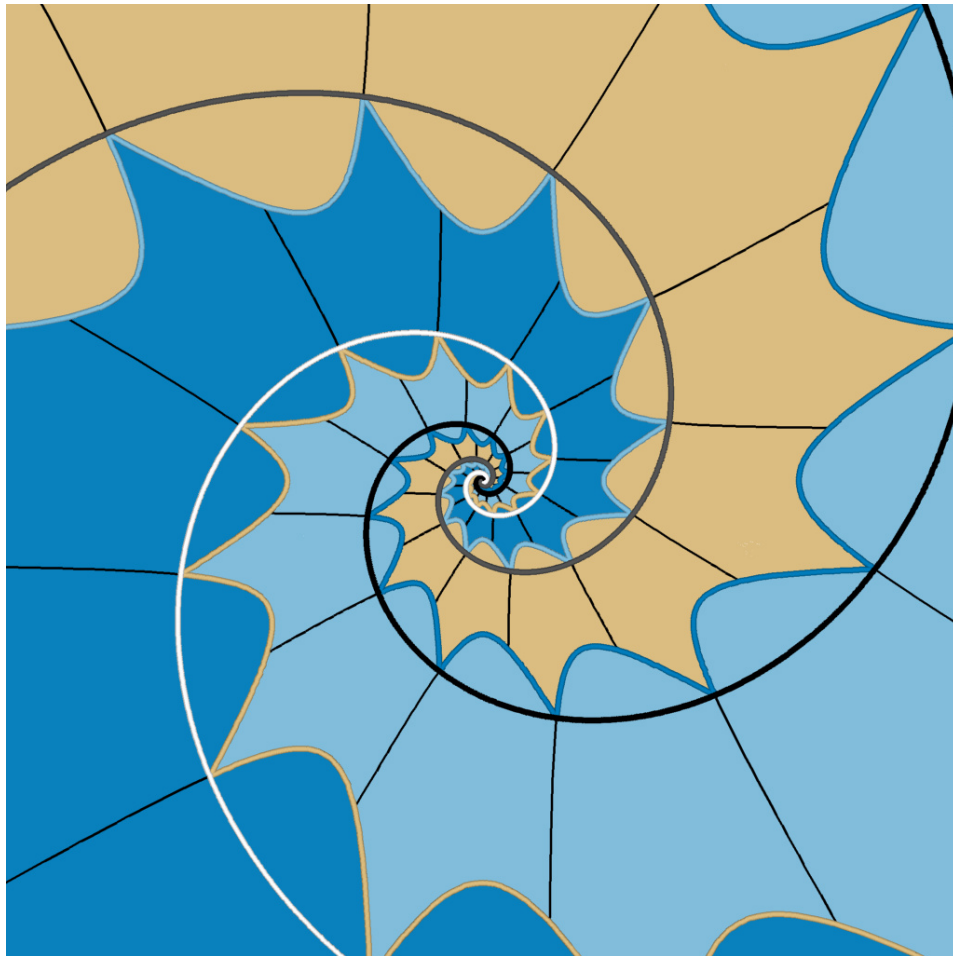
Figure 8: *"Seahorses in Play". Transformation of the geometry from Figure 6a to an approximation of a logarithmic spiral that scales by ϕ every $\frac{1}{2} \cdot \frac{1}{3}$ part of a circle.*

of a logarithmic spiral that scales by ϕ every $\frac{1}{2} \cdot \frac{1}{2}$, i.e., every counterclockwise quarter turn. Therefore, the resulting object is an approximation of the golden spiral.

Note further, this reparametrization is not conformal, i.e., angles are not preserved. Therefore, the corners of the rectangles are not right angles anymore. This could be alleviated by going to the complex plane and considering the map $z \mapsto \sqrt{z}$ as reparametrization, which is conformal. See [8] for other artistic applications of complex maps. In our case, however, the map would need to be altered just like the polar coordinate reparametrization in order to keep the spiral intact.

The white space can be filled exactly by a copy of the resulting approximation of the golden spiral. Said copy has to be rotated by π to fit the white space perfectly. This results in an arrangement of two intertwined approximations of the golden spiral, as shown in Figure 6b.

However, choosing a squeezing factor of $\frac{1}{2}$ was an arbitrary choice in the steps performed above. In fact, we can reparametrize using any squeezing factor of $\frac{1}{n}$, with $n \in \mathbb{N}$. This, in turn creates enough white, empty space to fit $n - 1$ reparametrized copies of the original approximated "better golden spiral", provided appropriate rotations.

For our final piece, we have chosen $n = 3$. The three resulting intertwined spirals are given complementary colors from a maritime palette. These colors and the shapes of the distorted rectangles allude to tails of sea horses. The palette also refers to the habitat of the nautilus—which sadly comes with only one, not even golden, spiral. Find the digital image "Sea Horses in Play" in Figure 8.

Acknowledgments

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