Categorizing Celtic Knot Designs

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Abstract

In this paper we investigate and enumerate classes of rectangular Celtic knot designs. We introduce criteria for reducing, filtering, and categorizing such designs in order to obtain pleasing, Celtic-looking patterns. We use Hamming graphs to manage large collections of related designs, begin to uncover relationships between Celtic designs, and attempt to identify mathematically and aesthetically significant characteristics of designs.

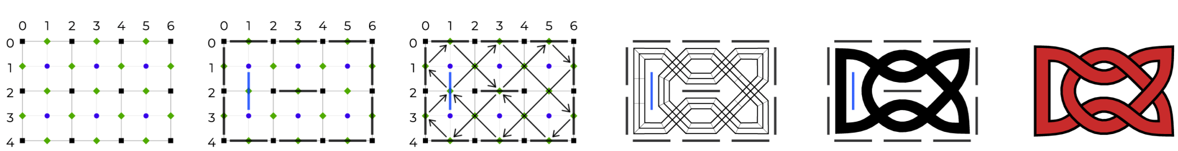
Introduction

What makes traditional Celtic knots beautiful? What structures and patterns do Celtic knots have in common, and how are different Celtic knots related to each other? Beyond style and hand, what makes a mathematical knot look traditionally Celtic? In this paper we attempt to answer these questions by characterizing, constructing, enumerating, and exploring spaces of Celtic knot designs.

Celtic knots and related structures have been studied widely in the mathematical literature: Cromwell explored symmetries of Celtic knot friezes [2]; Fisher and Mellor counted components for certain types of Celtic designs [4]; Lee and Ludwig have done extensive work classifying mosaic knots [8]; and Gross and Tucker have investigated knot polynomials of Celtic knots [6]. We build on this work by enumerating various classes of "Celtic-like" knot designs and using quotients of Hamming graphs to examine the relationships between closely-related Celtic designs.

Constructing Celtic Knot Designs

Celtic knot designs are often grid-based [2, 9]. To construct an $m \times n$ Celtic design, we use a 2-dimensional "doubled-up" array of points $\langle c, r\rangle$, where $0 \leq c \leq 2m$ and $0 \leq r \leq 2n$. This array contains three subgrids: The main grid consists of the points $\langle c, r\rangle$ where c and r are both even (shown as black squares). The second grid consists of the points $\langle c, r\rangle$ where c and r are both odd (shown blue circles). The remaining points, where $(c+r)$ are odd, are the ones that forms the basis of the Celtic design itself (shown as green diamonds). We call these **pivot points** (called construction dots [3]), because this is where we may insert **breaks** that connect adjacent grid points on either side. These three grids are referred to as the primary, secondary, and tertiary grids, respectively [9]. The choices of vertical, horizontal, or absent breaks at these pivot points completely determines a unique Celtic design, as illustrated in the following sequence of 3×2 designs:



We refer to break lines between points on the "even" grid (connecting black squares) as **type-0 breaks**, and those between points on the "odd" grid (connecting blue dots) as **type-1 breaks**. Each pivot point can therefore be the center of a type-0 break, a type-1 break, or no break at all. In this paper, we restrict our attention to *closed* designs where all the pivot points along the boundary have type-0 breaks. Otherwise, we would have to consider designs where the curves are allowed to "escape" the design.

To construct a design, imagine that you start at one of the pivot points and shoot a billiard ball at a 45 degree angle towards the center of the design. Whenever you encounter a break, you "bounce off" the break, possibly straightening, smoothing, or stylizing your curve along the way to form a pleasant-looking path. Because of this "bouncing" construction, such curves are also called *mirror curves* [5]. After bouncing around for a while, you eventually end up where you started, forming a design component we call a **band**. Repeat for any pivot points that you have not yet visited until you cover all of the pivot points, to obtain a completed Celtic design of one or more bands. In the above example, there is a vertical type-1 break on the left side and a horizontal type-0 break in the middle, resulting in a single-band design.

Equivalence Classes of Celtic Designs

We begin our journey by considering the most obvious question: *How many different Celtic designs are there, given certain dimensions and restrictions?* First, it is natural to identify designs that are equivalent up to rotations and reflections. In terms of group theory, this means looking at the equivalence classes, or the orbits, of designs under the actions of the dihedral groups D_4 and D_2 for the square and non-square rectangular designs, respectively. Another initial restriction, which in our experience gives rise to nice small designs, is to require all the non-boundary breaks to be type-1 breaks. We call such designs **type-1 designs** (similarly for **type-0 designs**). In general, a design could have type-0 breaks, type-1 breaks, both, or neither.

Let us begin with 3×2 type-1 designs. In this case, there are exactly seven pivot points, where there could either be a type-1 break or no break, and therefore $2^7 = 128$ designs in total, partitioned into the 48 equvialence classes shown in Figure 1. There are many identifiable properties of such designs: Some have multiple bands, some are **connected** (meaning that you cannot separate them into two pieces with a closed curve), and some have undesirable redundancies. As we continue through the paper we will introduce these properties and make aesthetic decisions about which designs to include.

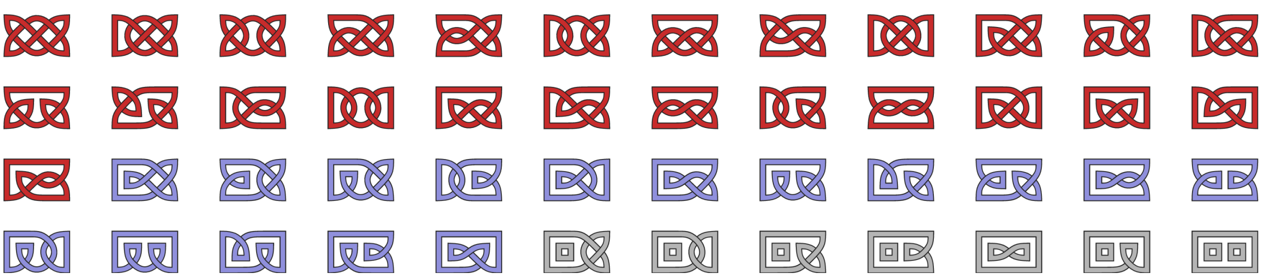


Figure 1: *The* 48 *equivalence classes that are* 3×2 *and type-1;* 41 *of which are connected.*

The restriction to type-1 designs works well for small numbers and is highly recommended when starting out exploring and drawing Celtic knots. *You should put away this paper now, and see if you can find all the* 48 *essentially different* 3×2 *type-1 designs on your own.* (Another great starting point is to explore the $m \times 2$ type-1 designs.) For those who wish to prove that there are exactly 48 designs of this type, here is a brief proof: By Burnside's Lemma, the number of essentially different 3×2 type-1 designs is the same as the average number of those designs fixed by one of the four symmetries of D_2. Designs that are fixed by reflection over the vertical axis are determined by break information at just four of the pivot points; for reflection along the horizontal axis, by five pivot points; and for 180-degree rotation, by four pivot points. All $2^7 = 128$ designs are fixed by the identity element of D_2. Thus, by Burnside's Lemma there are $(2^4 + 2^5 + 2^4 + 2^7)/4 = 48$ essentially different 3×2 type-1 designs, up to D_2 symmetry, as shown in Figure 1. For the remainder of this paper we will refer to equivalence classes of designs as simply "designs."

We are of course also interested in designs with more than one type of break, which are prevalent in traditional Celtic art. Each of the seven pivot points can either have a type-0 break, or a type-1 break, or none, and thus there are $3^7 = 2817$ possible 3×2 designs, partitioned into 648 equivalence classes (see Figure 2). This is a large space of designs, even though m and n are very small! Our next step will be to eliminate certain less desirable design configurations so that we can investigate more manageable collections and begin to visualize how different designs are related to each other.

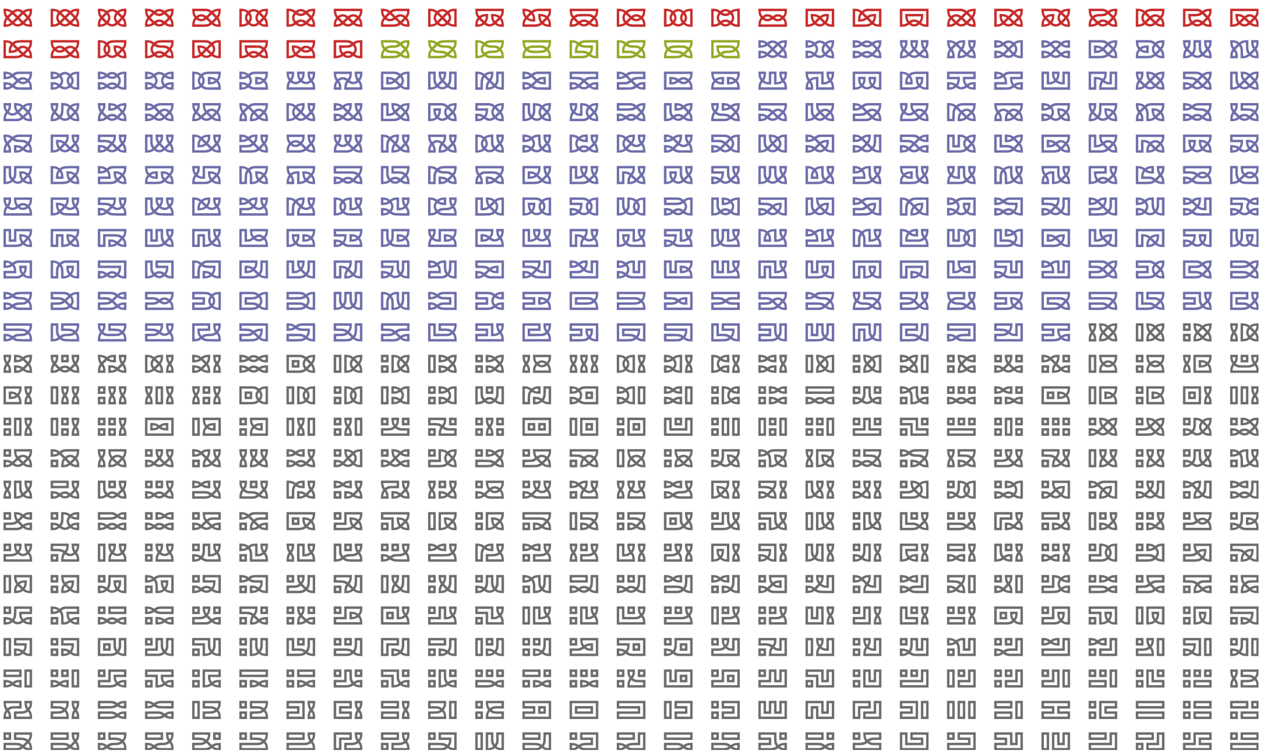


Figure 2: *There are* 648 *essentially different* 3×2 *designs;* 293 *of these are connected.*

Identifying the "Best" Designs

Even in the 3×2 case we can start to see conditions that we would like to impose. Informally and intuitively, we are after the designs that are the most *aesthetically pleasing* and *Celtic-looking*.

A significant property is whether or not a design has a *tail*. A design has a **tail** exactly when at least three breaks start forming a square. For example, in Figure 1, the last 23 of the 48 designs (colored blue and gray) have tails. A tail is part of a band that has one of three shapes: A *donut* (caused by four breaks in a square,), a *twist* (caused by three breaks of a square, , in any orientation), or a *sink* (where exactly one break in a square has been flipped, , in any orientation). This definition of a tail is similar to, but not exactly the same as, the one used in [3].

The restriction to connected and tail-free designs allows us to reduce our space of 3×2 designs significantly. Out of the 648 essentially different 3×2 designs shown in Figure 2, we can discard 355 *disconnected* designs (colored gray), and another 250 designs with a *tail* (colored blue). This leaves us with 43 connected and tail-free designs. However, some of these designs are still not very aesthetically pleasing.

Can you spot which? We propose that eight of the designs in the middle of the second row (colored green) should also be removed from consideration. Each of these is a "stretched-out" version of an equivalent 2×2 design (and can thus be compressed into 2×2 designs). These designs have what we call **ladders**: consecutive breaks alternating between type-0 and type-1 that extend all the way from the top to the bottom (or from the left to the right).

Figure 3: *Eight connected and tail-free* 3×2 *designs with ladders (marked in red).*

The space of 3×2 designs has thus been reduced to just 35 designs. We call these *reduced*. A design is **reduced** if it is *connected*, *tail-free*, and *ladder-free*. Reduced designs are pleasing to the eye in their efficiency and fullness. Reduced designs also tend to have a sufficient amount of crossings to be interesting (as suggested in [3]).

We have thus identified several useful criteria for finding aesthetically pleasing and Celtic-looking designs. In particular, we have utilized the notions of being *tail-free* and *ladder-free*. However, we are of course *not* saying that all aesthetically pleasing and Celtic-looking designs are tail-free, ladder-free, and connected. There are notable exceptions, like the *Six knots series* of prints by Albrecht Dürer after Leonardo da Vinci, in which tails are prevalent.

What further restrictions could be imposed? One obvious possibility is to reduce our attention to **single-band** designs (also called *mono-linear* in [5, 7]). There is one more condition that we could impose, and that is the essential artistic element of **symmetry**. For example, out of the 19 single-band and reduced 3×2 designs in Figure 4, there are nine designs with rotation or reflection symmetry. We have thus partitioned the 2187 possible 3×2 designs into 648 equivalence classes and further filtered them into, in this case, 19 nice, and essentially different, Celtic patterns, nine of which are symmetric.

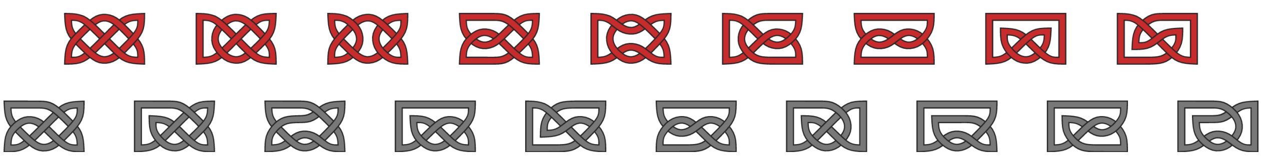


Figure 4: *The* 19 *essentially different* 3×2 *reduced, single-band designs, nine of which are symmetric.*

For even slightly larger values of m and n, there is a combinatorial explosion of possible designs, but our filtering methods will allow us to reduce to collections of manageable size. For example, for 4×3 designs, there are 17 pivot points, and thus $2^{17} = 131,072$ possible type-1 designs, which partition into $33,408$ equivalence classes (and $3^{17} = 129,140,163$ possible designs, which partition into $32,319,486$ equivalence classes). If we filter the type-1 designs to include only those that are *reduced*, *single-band*, and *symmetric*, we are left with the 184 designs in Figure 5.

Exploring the Space of Celtic Knot Designs with Hamming Graphs

Now that we can filter the space of desirable Celtic designs to more reasonable sizes, we can start to explore how the designs themselves are related. *How can we navigate the space of designs in a good way?* One way is by representing each design as a vertex in a graph, and connecting two designs with an edge when they differ by exactly one break.

Let us begin with a very simple example: The complete space of 2×2 Celtic designs. These designs

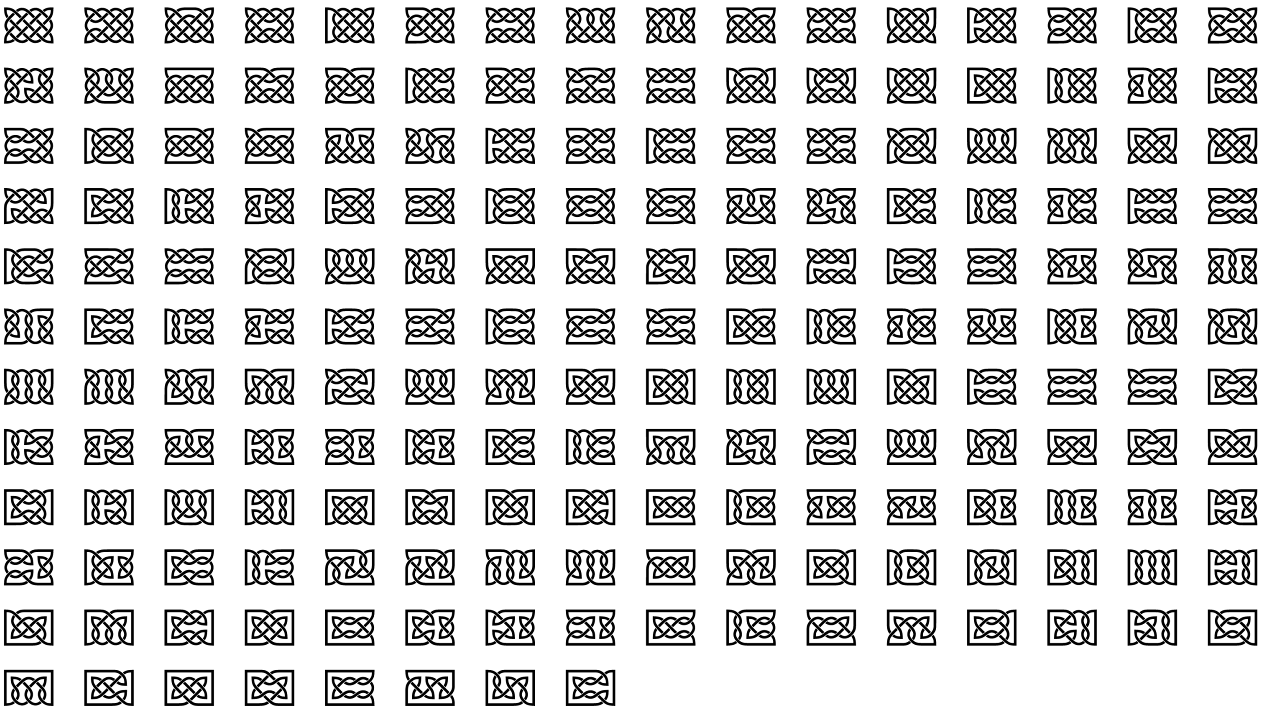


Figure 5: *The* 184 *designs that are* 4×3*, type-1, reduced, single-band, and symmetric.*

have four pivot points, and therefore at the outset there are $3^4 = 81$ possible such designs, illustrated in the left-most graph in Figure 6. Some of these designs are equivalent under rotation and reflection, and these *cliques* are shown connected with blue edges in the second graph. In the third graph, we have contracted the blue edges so that equivalent designs are grouped together into 21 equivalence classes (each of size 1, 2, 4, or 8). In the fourth image, the designs themselves are drawn in front of the vertices. Finally, by choosing one representative for each equivalence class, we obtain the fifth graph. Here, each vertex is now an equivalence class of designs, and two equivalence classes are connected with an edge if at least one of the designs in each class differ by just one pivot point.

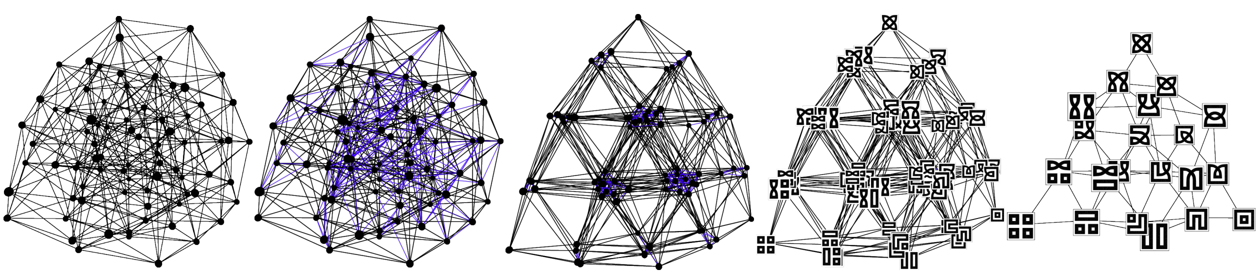


Figure 6: *Hamming graphs for* 2×2 *designs.*

The first graph above is also known as the *Hamming graph* $H(4, 3)$ on the set of ordered 4-tuples of elements from $\{0, 1, 2\}$, where each vertex is a choice of breaks on the four pivot points, and edges connect vertex tuples that differ in exactly one coordinate. The final graph is the quotient of the Hamming graph under the equivalence relation that relates symmetric designs. For simplicity we will also refer to this quotient type of graph as a **Hamming graph**.

By looking at these Hamming graphs for collections of Celtic designs, we can identify interesting relationships between designs, and pick out key designmatic elements. In addition, the Hamming graphs of design collections have geometric structures that are interesting to explore. For example, in the 2×2 design collection pictured above, we see a triangular structure, because there are *three* possibilites for each pivot point. Figure 7 shows a similar progression, from the 128 different 3×2 type-1 designs to the 48 equivalence classes identified in Figure 1.

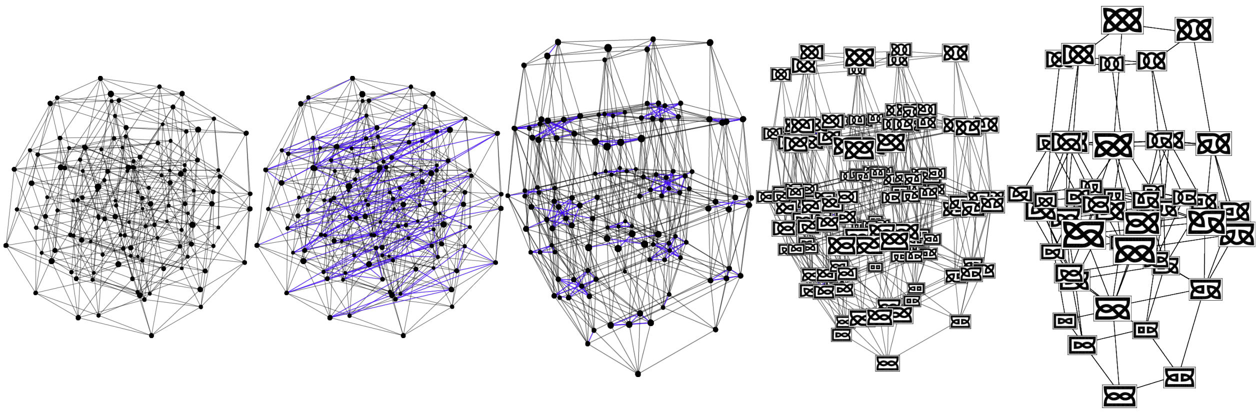


Figure 7: *Hamming graphs for* 3×2 *type-1 designs.*

Let us now expand our view to some of the larger collections of designs we discussed earlier. We saw in Figure 2 that there are 648 essentially different 3×2 designs. These are pictured in the leftmost Hamming graph in Figure 8. The second graph shows the 43 connected and tail-free designs. The third is the Hamming graph for the 35 of those designs that are reduced. The fourth shows the 25 of these designs that are type-1. The colors indicate the degrees of the vertices (red means higher degree).

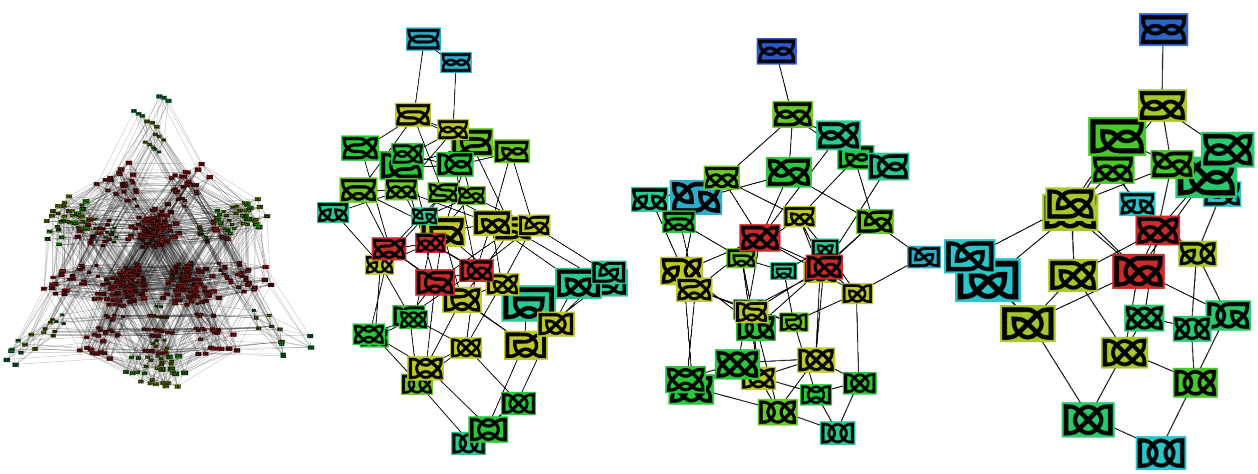


Figure 8: *Hamming graphs for* 3×2 *designs.*

We saw in Figure 5 that there are 184 equivalence classes of reduced, single-band, symmetric 4×3 type-1 designs. Without the additional condition of symmetry, there are 2533 such designs, shown in the two leftmost birds-nest style Hamming graphs in Figure 9. After imposing the symmetry condition we obtain the highly disconnected graph shown on the right side with and without designs on top of the vertices. It makes sense that Hamming subgraphs for symmetric collections of designs are so disconnected, because changing a choice of break at any pivot point is likely to remove at least one of the symmetries.

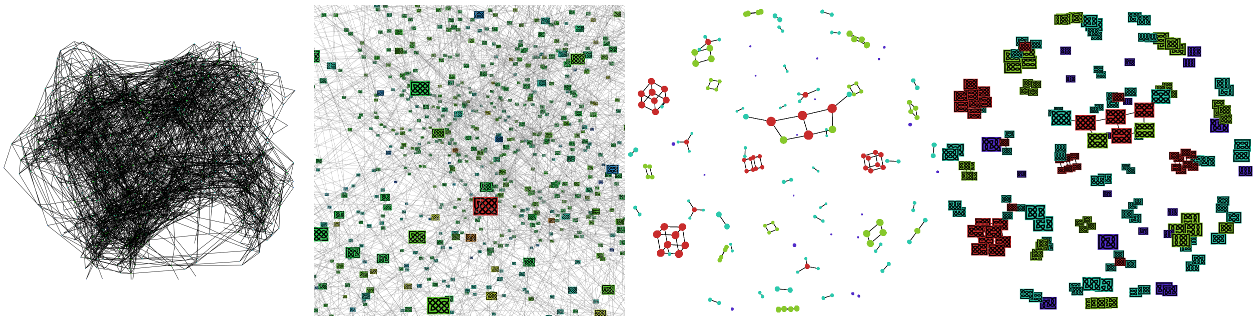


Figure 9: *Hamming graphs for* 4×3 *type-1, reduced, single-band designs.*

Conclusions and Future Work

Many Hamming graphs have striking bilateral symmetries that we hope to study in future work. The leftmost graph in Figure 10 is the Hamming graph for the 175 reduced 3×3 type-1 designs; the second and the third graph show the 94 with a single band. *What causes the symmetries in these graphs?* We can also ask questions about connectivity in these graphs, for example the red design with high degree in the second graph of Figure 9.

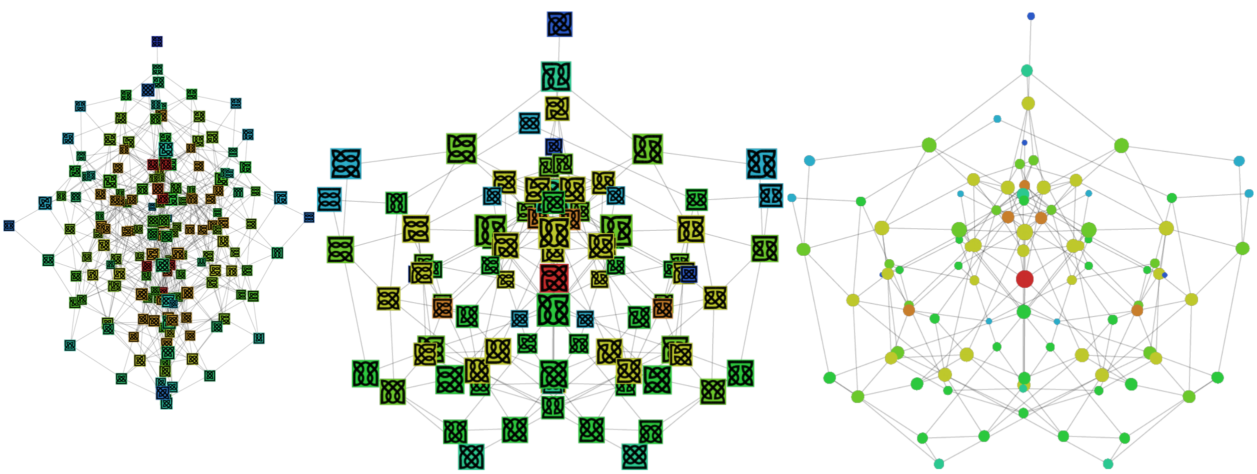


Figure 10: *Symmetric Hamming graphs for* 3×3 *type-1 designs.*

Furthermore, by investigating the global structure of the Hamming graphs of the various spaces, we are able to identify interesting designs, visually similar designs, and designs that for other reasons stick out. The fact the some of these graph are apparently without symmetries speaks to the somewhat tangled nature of the space of Celtic designs.

In this work we restricted our attention to reduced Celtic designs that fill rectangular regions, but there are many examples of traditional Celtic art based on non-rectangular regions and/or non-rectangular grids [9, 10, 1]. Future work could extend our enumeration visualizations to non-rectangular Celtic designs.

We can also use enumerations of Celtic designs to further the study of mosaic knots as described in [8], or, along similar lines, to determine "Celtic numbers" for knots based on the minimim size of their possible reduced rectangular Celtic designs. Note that we can use Celtic designs to study both alternating and non-alternating knots, because although each Celtic design can determine an alternating knot [6], we can also choose design crossings to create non-alternating knots. Such non-alternating Celtic knots have on rare occasions appeared in traditional Celtic patterns, for example the Govan Stone pattern illustrated in [2].

For larger values of m and n than we covered in this work, it is computationally prohibitive even to enumerate reduced collections of designs. However, we can use our filtering conditions and the software tool written by the first author to produce examples of particularly pleasant Celtic designs of any size. For example, we end with one hundred randomly generated 3×6 reduced designs with rotational symmetry.

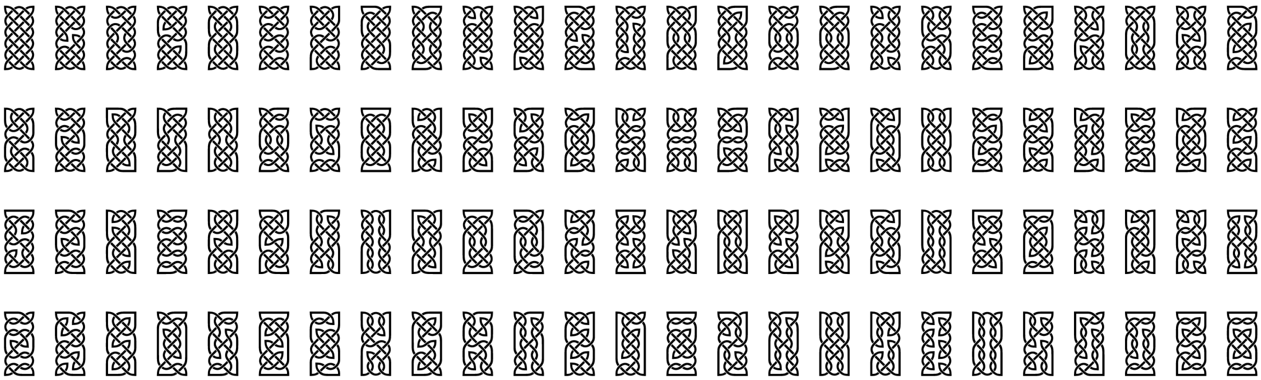


Figure 11: *One hundred randomly generated* 3×6 *reduced and rotationally symmetric designs.*

For updates on future work, higher-resolution images, collections of Celtic knot designs, and more, see https://rant.codes/celtic.

Acknowledgements

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