Dancing Topologically

Karl Schaffer[1]

[1]De Anza College and MoveSpeakSpin; karl_schaffer@yahoo.com

Abstract

This paper was initially motivated by efforts to develop video performance in the time of COVID that incorporates split screen clips of dancers virtually interacting with each other. This led to play with several topological surface patterns in which connections between side by side video clips explores connection in a time when people are necessarily separated. The attention to the paths of objects in these videos led to an examination of paths of various types in dance composition, which touch on several other connections between dance and topological structures, including floor paths, winding and turning numbers, and especially knots and links. This is an initial effort to examine several topological properties found in many dance forms.

Introduction

Topological properties arise in dance a number of ways, for example in the complex tangle of paths dancers take on a stage. This can be seen to relate to concepts such as knots, links, homotopy, and winding numbers. How things connect is at the heart of the mathematics of graphs or networks, which appear when we consider connections among dancers. We will look at several aspects of topological properties of dance, with an eye toward exploring further and inspiring interest by both dancers and mathematicians.

A number of dancers and mathematicians have explored topological properties of dance. Several papers have investigated knotting patterns of the ribbons produced in Maypole dance [8,12,16,17]; Andrea Hawksley also applied her analysis to braid patterns in certain traditional folk dances. Diana Davis's 2012 Dance Your PhD entry employs dancers' paths to demonstrate a mathematical result on the translation surface produced by identifying pairs of edges on a double pentagon [6,7]. Mike Naylor and Vi Hart investigated ways for 5 or more people to create star patterns when holding hands in a circle [15]. Scott Kim, Barbara Susco, and I created a show in 1995, "Through the Loop: In Search of the Perfect Square," in which we performed over 20 traditional and newly invented string figures as part of dances, theatrical interludes, and audience interactions, Figure 1 [25]. These employ the "unknot," or "trivial knot," a simple closed loop, in complex and often traditional manipulations. Many are demonstrated in a series of short videos by George Csicsery [5]. Erik Stern and I created a number of dance pieces with bungee cord loops in 1999 [21]. Aerial dancers, especially in the performance form known as silk, exploit many similar mathematical ideas. Nancy Scherich produced a dance video comparing aspects of geometry, algebra, and topology [26], and her Dance Your Ph.D. video explores braid groups with aerial dancers [27]. Hart, Hawksley, Segerman, Matsumoto and their collaborators have created games, videos, and written papers exploring the mathematics of unusual topological spaces, such as in [11].

Figure 1: *Ann Arbor Students; "Through the Loop" [25]; Schaffer & Stern: "Bounds of Discovery"[21]..*

Topology deals with properties of geometric objects that are preserved when the object is stretched, compressed or twisted, without tearing or breaking. For example, if two dancers grasp right hands and perform a sequence of movements without releasing the hold, then we might consider that sequence a continuous deformation in time of the overall shape of their two bodies – until they release hands, at which point the continuity disappears. In fact this particular connection – and disconnection – was at play in a vaudevillian handshake routine at the heart of one of the first mathematically oriented dances that Erik Stern and I created in 1990 [22]. It also led to a series of classroom activities in which we ask participants to count how many distinct ways there are for two or more dancers to connect via grasped hands; the distinctions participants come up with are essentially topological in nature [24, ch. 1].

Although videos record three-dimensional scenes, video clips are themselves two-dimensional artifacts. If we imagine gluing the right edge of a horizontal rectangular screen to the left edge so the arrows match, then an object passing off the left edge (known as stage right!) shortly reappears at the right edge (stage left), and we have essentially created a mathematical cylinder. If the upper edge is glued to the bottom edge so those arrows match, then either the surface known as the Klein bottle (Fig. 2b) or the torus (Fig. 2c) is created, depending on the direction of those edges' arrows.

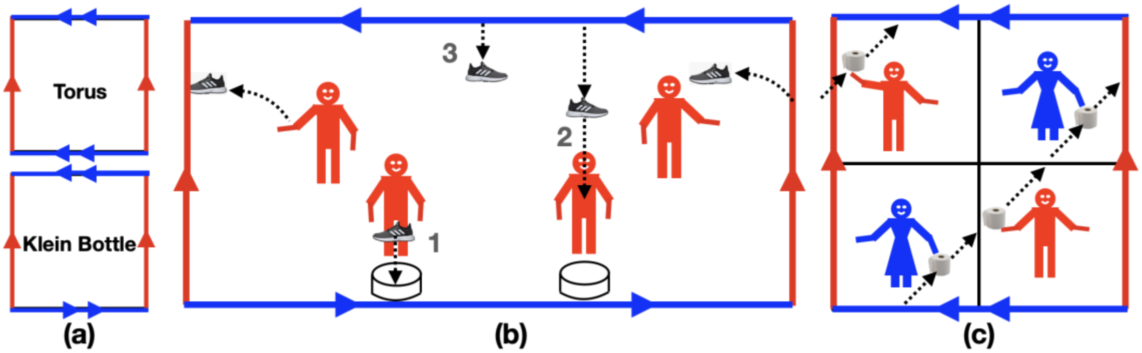

Figure 2: *Klein bottle and toroidal dances: (b) "Lost in Translation"[23], (c) work in progress.*

In 1993, Stern, Scott Kim and I created a dance work, "Lost in Translation," in one scene of which the stage acted like a Klein bottle [23]. As one of us tossed a tennis shoe offstage right, Fig. 2b, it seemed to reappear tossed to us from stage left. One of us then dropped a shoe into a bucket downstage right, Fig. 2b(1), whence it reappeared dropped to us from above stage left, Fig. 2b(2). However, having set up the expectation that the downstage edge connects in reverse to the upper edge of the proscenium, we got a laugh by next dropping the shoe into a stage left bucket, whence it plopped down to the floor center stage, Fig. 2b(3)! Although the stage is 3- rather than 2-dimensional, this sequence essentially created a "thick" model of the 2-dimensional Klein bottle. If one were to have performers disappear through the upstage curtain, only to reappear at the back of the audience, one would then have created a model of the 3-dimensional manifold known as $K^2 \times S^1$, the cross product of the Klein bottle and a circle [14, pg 136]. The trick of having performers disappear upstage only to reappear behind the audience has frequently been used by stage magicians, for example see the story on David Copperfield's use of this effect [1].

In the time of COVID many dancers and choreographers have been creating video performances by physically distant dancers, connected only by side by side video clips. Sometimes dancers appear to reach from their screen to that of a neighbor [18]. We have begun experimenting with the use of contiguous video clips to create a dance about human connection in a time of social distancing. For example, we are working on dance clips in which dancers appear to pass objects that have become symbols of this time from screen to screen, such as rolls of hoarded toilet paper! To date we have found the toroidal model most productive for this video dance work, but may experiment with other models, Figure 2c. However, the goal is not to display every possible type of surface, but to figure out how best to communicate the overall artistic intent of the work relating to dance in the time of social distancing.

In our video work the objects passed among dancers follow interesting and hopefully entertaining, or at least theatrically evocative, paths from one screen to another. But paths of dancers, usually in the plane of the stage, are also crucial in dance. This suggests the mathematical subject of homotopy, a concept used to establish the algebraic structure of topological spaces by examining how continuous paths may be deformed into each other by stretching and pulling, without breaking. In Figure 3a the solid clockwise circular path shows a path dancer x might follow which returns to x's starting point. Often the choreography simply demands that x follow this path only approximately, so that the dotted path might suffice as well. Most likely there may be time constraints keyed to music cues, for example x might be asked to execute the choreography within an 8-bar phrase. But a variety of paths might suffice, and we would say that the solid path is homotopic to the dotted path, since it may be pushed and pulled without breaking to generate the dotted path. Figure 3b shows a sequence of smaller and smaller clockwise circular paths that conceivably reduce the solid path to a null path in which x executes the choreography essentially standing in place. In this case the mathematical language states that the solid path is homotopic to a null path.

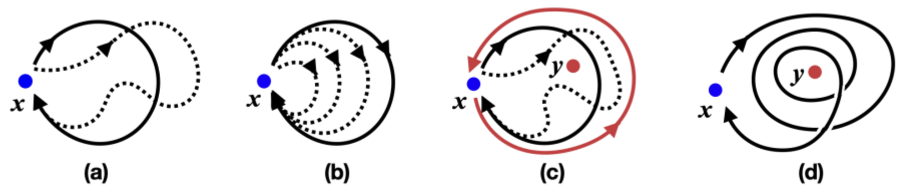

Figure 3: *Dance homotopy: (a),(b) paths of x homotopic to null path; (c),(d) x's paths around y.*

However, Figure 3c shows a second dancer, y, whom x performs a clockwise circle around before returning to the starting point. As long as x and y start in different positions – and y does not move! – there is no way for x to execute the choreographic requirement by shrinking the circle down to nothing. The red counter-clockwise circle shows a path for x around y that is the reverse of the solid clockwise black path. In mathematical terms, it undoes the black path and is its negative, and the composition of the two paths would be considered equivalent to a null path for x that does not move from x's starting position. However, for the dancers the black path followed by the red is not equivalent to x simply doing the choreography in place! Though the red and black paths are mathematically opposite because they circle in opposite directions, choreographers might find the two practically equivalent. Figure 3d shows x following a clockwise path three times around y and finally returning to the starting point. In the mathematics of homotopy, the algebraic structure of x's paths in relation to y is that of the integers, in the sense that any number of positive (traditionally counter-clockwise) or negative (clockwise) circles around y are possible, and these circles compose, mathematically at least, in the same way that integers add (see for example [14]). Although it might seem that homotopy is of limited usefulness in understanding choreographic patterns, when more dancers are added to the mix, it might help in modeling the patterns that arise.

In Figure 3d the last portion of the path is drawn to appear above the earlier sections, as if the dancer had trailed a string behind so as to trace out the path on the floor. This type of model was used by Andrea Hawksley to analyze the dance Waves of Tory in [12], though she employed a model in which earlier sections of a path are drawn above later sections. Instead, we draw earlier path sections so they pass below at crossings, so as to coincide with how the strings might appear from above if strings are actually used, as in Hawksley's workshop, and we use this to model dance paths in general. We also adopt the proviso that at any given point where two sections of a path meet, they actually do cross, with the path section below erased just around the junction. Further, no more than two sections of a path are allowed to cross at the same point. If dancers' paths fail to obey these constraints, slight adjustments to the paths will often fix the problem without altering the mathematical or choreographic properties of the dance.

A number of mathematical questions arise. For example if a solo dancer follows a path on a flat surface that eventually returns to its starting point, Figure 4a, might this modeling technique trace out a knot, or will it always trace the unknot? We can see intuitively why the drawing of the path must always be the unknot, [4, pg 55], by imagining that the path rises continuously from time t = 0, until it reaches a point above the starting point, and then drops down to the initial point at time $t = 1$. Looking at the path from the side, as in Figure 4b, we can see the path winding upwards around the dotted line much as a single strand of ribbon winds back and forth around a maypole. The dotted line back to the start is slightly tilted from vertical so that none of its points fall above or below other points of the diagram. We can see intuitively why this might produce a number of windings around the pole, but no knot, by sliding each point of the rising strand horizontally away from a vertical axis line rising above the starting point, so that the higher the point is, the farther from the axis it is. At the top the strand than passes across the top to the tilted dotted line, which takes it back down to the start. If we now re-project the knot to the horizontal plane, we get a diagram like Figure 4d showing windings around the starting point with the only crossings now the strand across the top. This can be unwound to the unknot loop using what are called Reidemeister moves I and II, described later in this paper.

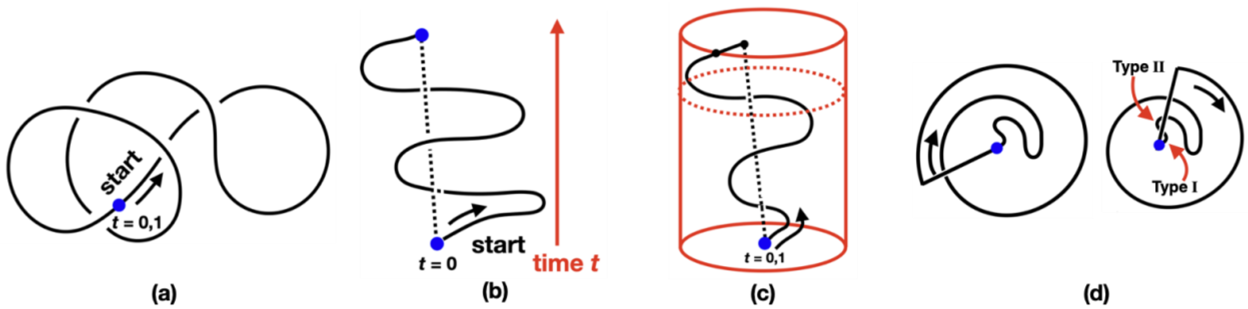

Figure 4: *The path of a solo dancer that returns to the start always traces the unknot.*

We might also ask if every connected unknotted closed path is "danceable," meaning it might be the floor path of an actual solo dancer who observes the over/under drawing rule that every overcrossing is only performed in time after its corresponding undercrossing. If the path has two or fewer crossings, then it will always be danceable. Fig. 5a shows two possible types of drawings corresponding to a path with two crossings: BABA for Below-Above-Below-Above, and BBAA for Below-Below-Above-Above. In both cases we may consider the four letters as part of a cycle, so that, for example BBAA is identical to BAAB. BBAA may be performed with the two initial Below segments prior to the Above segments, so is danceable. And BABA must have the Below segment next to its corresponding Above segment, either going forwards or backwards in the cycle, so is also danceable, as after that BA will be another B and then it's corresponding A. Note that the BABA path is danceable in both directions, the BBAA path in only one direction. If the path drawing has three or more crossings, it might not be danceable, for example see Figure 5b.

Since knot drawings are not danceable by a solo dancer, we might ask whether they may be duet- or 2-danceable by two dancers. In this case we want the two dancers to start independently at different points, move along the path in the same direction, and end at the other dancer's initial point, presumably at the same time, always following the over/under crossing rule. Both might aim to complete their sections of the path in an identical musical phrase, even though each might vary their speed at any point. Fig. 5c shows a duet-danceable plan for the 3-crossing diagram of Fig. 5b. Fig. 5d shows that the same figure is what we might call reverse 2-danceable. The two dancers begin at the same point and travel away from each other along the path, arriving at a common point after together traveling along the entire figure.

We might define n-danceable knot drawings similarly. Fig. 5e shows an unknotted 2-crossing figure for partners x_1 and x_2, y_1 and y_2, that is 4-danceable and which is used in the "straight line hey for four" sequence common in contra and English Country dance. Fig. 5f shows a circular hey for four figure also common in these traditional folk dance forms. Mathematically it is known as the (2,4) torus link, the 2 indicating it has two component loops that are "linked," the 4 that it may be drawn with 4 crossings. It is a torus link because it can be drawn on the surface of the torus without any crossings. It is also known as one of the smallest Celtic knots or links. For connections between the hey for four, n-body choreographies in physics, and change-ringing, see [19].

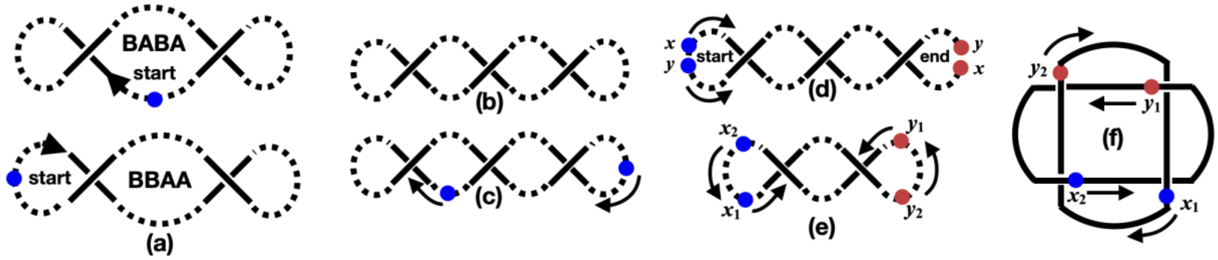

Figure 5: *(a) Any 2-crossing diagram is always solo danceable. (b) A three crossing unknotted diagram that is not solo danceable, but is duet danceable (c) and reverse duet danceable (d). (e) The "straight line" 4-danceable hey for four. (f) The circular hey for four figure is the 4-danceable (2,4) torus link.*

Every *n*-crossing knot drawing is clearly *n*-danceable, since we may assign one dancer to the underpass of each crossing, all moving in the same direction along the knot. The only alternating knot drawings that are reverse duet-danceable are those that have the form of Figure 5d, an unknot composed of *n* twists, all twisted in the same direction. Fig. 6a shows why this is true. Dancers *x* and *y* begin moving towards crossings C_1 and C_2 respectively. Dancer *x* is stopped at z_1 at the overpass of crossing C_1, and *y* is blocked at z_2 at the overpass of the next crossing C_3 – unless crossing C_2 is actually identical to C_1, as in the diagram in Fig. 5d. This pattern continues at the following pair of crossings, which again must be identical if *x* and *y* are not to be blocked, and so on. Finally *x* and *y* bump into each other when the knot has been traversed. Fig. 6b shows such a figure, which is also the straight line hey for seven. Such figures with *n* crossings are the diagram for a hey for $n + 2$ dancers in contra or English Country Dance.

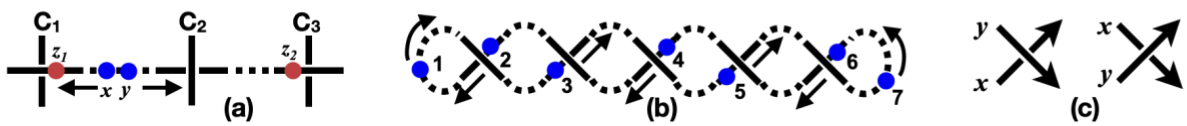

Figure 6: *(a) The only alternating knots that are reverse duet-danceable are a series of twists in the unknot, such as Figure 5d or (b), which is also a diagram for the straight line hey for seven dancers.*

Fig. 6c displays an important distinction worth mentioning when two dancers approach and perform a crossing close together while both face the direction of their own movement path (so do not perform turns around their own centers!) If the under crossing dancer *x* starts to the right of the over crossing dancer *y*, then *x* passes with *y* on *x*'s left, *y* sees *x* pass in front from right to left, and then *y* passes with *x* on the left – often called "passing left shoulders" in square dance or similar forms. If *x* starts to *y*'s left, then *x* passes with *y* on the right, and *y* then passes *x* on the right. Dancers and choreographers in all dance forms strive to keep careful track of such distinctions during rehearsal and performance!

Knots and links actually represent 3-dimensional objects that are typically shown in the plane via projections or drawings. Any knot will have more than one such drawing; for example, Fig. 5 displays several different drawings of the unknot. Two knot drawings may be shown to represent the same 3-dimensional knot if either may be transformed into the other by a sequence of the three "Reidemeister moves," shown in Fig. 7 [14, pg 56]. Although the Reidemeister moves preserve the identity of the 3-dimensional knots (or links) that they represent, unfortunately only move III preserves danceability. This can be seen by observing that the knot's initial available directions of traversal, shown by the vertical arrows, for types I and II differ before and after the move is performed; this is not the case for type III. This fact about type III might be important if a dancer appears to cross directly over or under a crossing already established by two other strands of a dance's diagram; in notating the dance, the third line could be slightly moved to either side of the existing crossing without affecting the knot diagram's traversability.

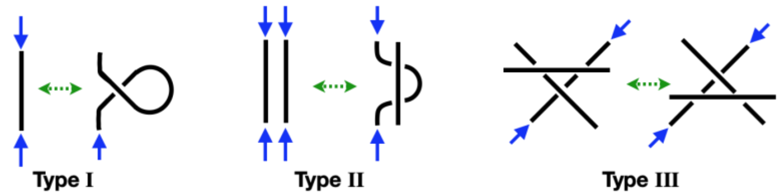

Figure 7: *Reidemeister type I and II moves alter the possible traversal directions, type III does not.*

Fig. 8a shows a trefoil knot drawing that is duet-danceable. Adding three type I twists in the same direction, Figure 8b, produces a trefoil drawing that is not duet-danceable (fewer type I twists do not accomplish this.) Figure 8a is an alternating knot drawing not of the form of Figure 5d so, is not reverse duet-danceable. Figure 8c shows an alternative approach to duet-danceability. Both dancers begin at an over-crossing and move away from it. This therefore leaves out a single point, breaking the knot diagram into a long string, so we will define this as a "string duet-danceable" design.

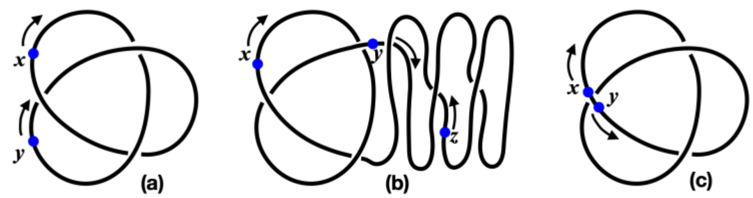

Figure 8: *(a) Duet-danceable diagram for a trefoil knot drawing. (b) Trefoil drawing that is not duet-danceable. (c) "String duet-danceable" diagram of the trefoil knot.*

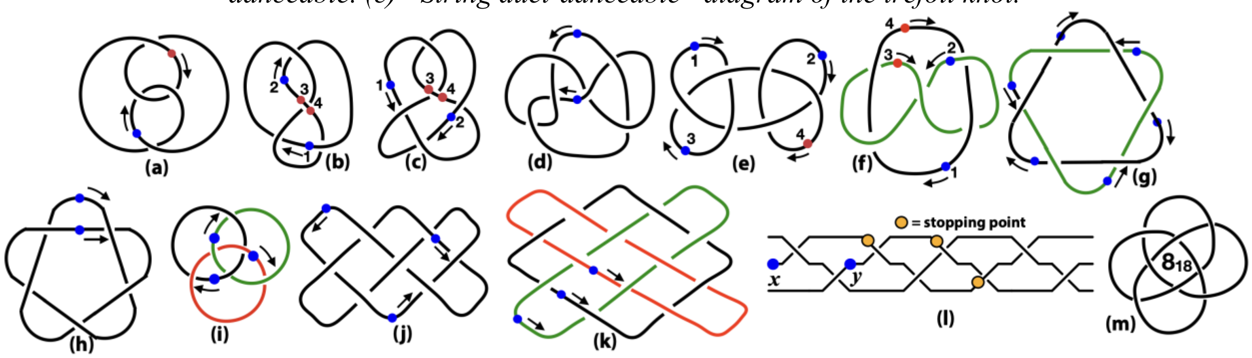

Figure 9: *Variety of links and knots and their danceability.*

Figure 9 shows danceable patterns for a variety of knots and links, most with 6 or fewer crossings:

(a) and (b) Two minimal crossing drawings of the 4-crossing figure 8 knot, each of which is duet-danceable. Dancers 3 and 4 in (b) show a string duet-danceable plan.

(c) The 5-crossing knot 5_2 is duet-danceable by 1 and 2, and string duet-danceable by 3 and 4.

(d) The -6 crossing knot 6_3 is duet-danceable.

(e) This "composite knot" called the granny knot is minimally 3-danceable by 1, 2, and 3, though 1, 2, 3, and 4 can perform a nicely symmetric 4-danceable pattern.

(f) 5-crossing Whitehead link, duet-danceable by 1 & 2, symmetrically danceable by 1,2,3, and 4.

(g) The circular hey for six is a toroidal alternating link in which dancers move in opposite circles and pass each other alternately on the right and left.

(h) A toroidal knot similar to (g), with an odd number of crossings is duet-danceable.

(i) The three component Borromean Rings is 3-danceable.

(j) and (k) 3-danceable Celtic knot and 3 by 3 Celtic link. All *n* by *n* Celtic links are *n*-danceable.

(l) and (m) This braid is knot 8_{18} and shows that some *B*-strand braids are not *B*–1-danceable. This is also the 3-person weave in basketball, e.g. the Golden State Warriors often use it in actual games [13].

For more examples of low-crossing number prime knot and link danceability see the supplement to this paper in the Bridges Archives.

Winding and Turning Numbers

The rehearsal video clip [20] includes duet dance phrases for the figure 8 knot and the link in Fig. 9f. In each case the dancers execute turns around a central point, around each other, and around themselves – and all of these may be different in number and direction. The number of turns of a dancer around a fixed central point is called the winding number. If we imagine a unit vector pointing in each dancer's forward direction, then the number of times that vector rotates is often called the turning number; here we simplify by assuming this vector always stays parallel to the plane of the stage, e.g. dancers won't roll on the floor!

Duet partner turns are very common in many world dance forms. In one pattern two dancers turn around each other as they circle the dance space. Fredrich A Zorn delineated the four possible patterns shown in Figure 10 [29]. Maria Basile, co-founder and co-director of the San Jose dance company SJDanceCo, an experienced social dance teacher, and frequent guest artist with our company, has explained how these circular paths are used in standard social dance forms [2]. In the standard waltz pattern (a) is called natural and (b) is known as the reverse form, whereas patterns (c) and (d) are usually not used. Maria says that (c) and (d) are possible in the tango, in which case they are known as moving backwards. She adds that in more free-form choreography any of these patterns as well as other floor paths are acceptable,

The pattern in Fig. 10a and b, known as a hypocyloid, is the locus of points on a circle that rolls inside a larger circle. Fig. 10c and d is an epicycloid, formed by rolling a circle outside another circle. Pattern (a) is perhaps most easily glimpsed in the Swedish partner dance Hambo, in which dancers turn around a common center in a smoothly continuous fashion [28].

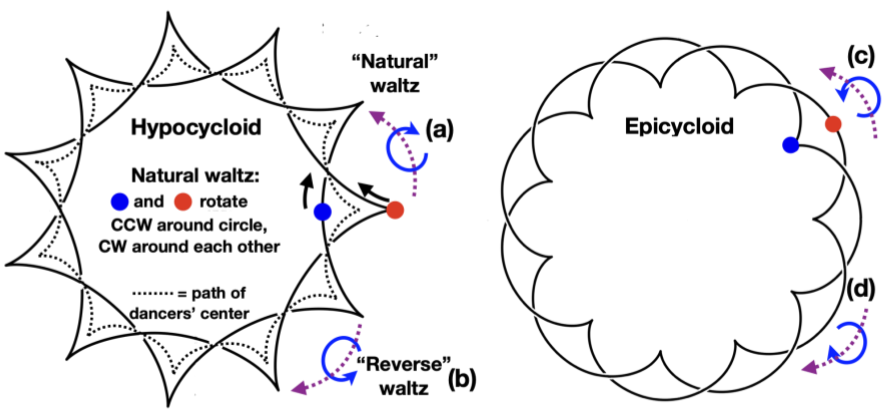

Figure 10: *Four circling patterns for partners turning around each other and the space.*

Future Work

Many interesting problems might be investigated further. Every knot has a braid representation, and its braid with minimal number of strands B clearly would be B-danceable. But how does danceability compare for knots and their braid or tangle forms? Can we characterize the unknots with n crossings which are solo-danceable? Find a good algorithm for the least n for which a minimal drawing of a prime knot is n-danceable? Decide which dance forms or choreographers tend to use which knot diagrams? Learn if algebraic simplifications of knot danceability are possible? Explore knot or link patterns when dancers connect by grasping hands or passing objects hand to hand? See how homotopy, winding, and turning numbers are embodied in dance? Determine whether attention to these topological characteristics have any aesthetic implications, or might help inform dancers, choreographers, audiences, or scholars? Creating dance phrases to match a knot or link, or mapping dance phrases as knots, might also make enjoyable math and dance class activities! A number of these ideas will be considered in a follow-up paper.

References

[1] T.M. Andrews. https://www.washingtonpost.com/news/arts-and-entertainment/wp/2018/04/18/a-david-copperfield-trick-allegedly-injured-a-participant-its-secret-was-just-exposed-in-court/. Washington Post. Apr. 19, 2018.

[2] M. Basile. Personal communication, Aug. 9, 2020.

[3] Cambridge University English Country Dance Club. https://round.soc.srcf.net/dances/cdb/cdb6/hey.

[4] P. Cromwell. *Knots and Links*. Cambridge University Press. 2004.

[5] G. Csicsery. Short films on string polyhedra. http://mathdance.org/html/CsicseryShortFilms.html.

[6] D. Davis. "Cutting Sequences on the Double Pentagon Explained Through Dance," 2012. https://vimeo.com/47049144.

[7] D. Davis. "Cutting sequences, regular polygons, and the Veech group." 2017. https://arxiv.org/pdf/1708.09552.pdf.

[8] S. Gerofsky, E. Knoll, and J. Forren. "Dancing Rope and Braid Into Being: Whole-body Learning in Creating Mathematical/ Architectural Structures." Bridges. 2017, pp 523-530. https://archive.bridgesmathart.org/2017/bridges2017-523.pdf.

[9] S. Gerofsky and T. Cove. "The Geometry of Longsword and Rapper Sword Locks." Bridges, Banff, 2009, pp 383-384. http://archive.bridgesmathart.org/2009/bridges2009-383.pdf.

[10] G. Hart. "Simons Foundation mathematical impressions: Longsword dancing." 2014. https://www.simonsfoundation.org/multimedia/mathematical-impressions-multimedia/mathematicalimpressions-long-sword-dancing/.

[11] V. Hart, A. Hawksley, E. A. Matsumoto, and H. Segerman. "Non-euclidean virtual reality I and II: explorations of H2 × E and H3. Bridges, Waterloo, 2017, pp 33-40 and 41-48. https://archive.bridgesmathart.org/2017/index.html.

[12] A. Hawksley. "Exploring Braids through Dance: The "Waves of Tory" Problem." Bridges, Towson, 2012, pp 613-618. https://archive.bridgesmathart.org/2012/bridges2012-613.html.

[13] How U. "Warriors Weave." 2015. https://www.youtube.com/watch?v=DEyULvNXBmo.

[14] R. Messer and P. Straffin. *Topology Now*. The Math. Assoc. of America. 2006.

[15] M. Naylor and V. Hart. "Human Geometry Workshop." *Bridges Proceedings*, 2011, pp 649-656.

[16] C. von Renesse and J. Campbell. "Mathematics and Maypole Dancing." Bridges. 2019, pp 615-620.

[17] C. von Renesse and J. Campbell. "Learning to love math through the exploration of maypole patterns". April 2019. Journal of Mathematics and the Arts 13(1-2):131-151.

[18] J. S. Rioux and E. Murray. "Thao & The Get Down Stay Down – Phenom." Dance video, 2020. https://www.youtube.com/watch?v=DGwQZrDNLO8&feature=emb_logo.

[19] K. Schaffer. "Dances of Heavenly Bodies Dance, N-body Choreographies, and Change Ringing." *Bridges Conference Proceedings*, Enschede, the Netherlands, Jul. 27–31, 2013, pp 543-546.

[20] K. Schaffer. "Knot and Link Floor Patterns." 2021. https://vimeo.com/manage/videos/543554623

[21] K. Schaffer and E. Stern. *The Bounds of Discovery*. Dance performance, 1999.

[22] K. Schaffer, E. Stern. *Dr. Schaffer and Mr. Stern: Two Guys Dancing About Math*. Dance performance, 1990. Video clip at https://www.youtube.com/watch?v=aVsZdx7cTjc, starts at 1:22.

[23] K. Schaffer, E. Stern, S. Kim. "Lost in Translation." Dance performance, 1993.

[24] K. Schaffer, E. Stern, S. Kim. *Math Dance*. MoveSpeakSpin. 2001.

[25] K. Schaffer, S. Kim, B. Susco. *Through the Loop*. Dance performance, 1995.

[26] N. Scherich. "Algebra, Geometry and Topology: What's The Difference?" 2019. https://nancyscherich.com/algebra-geometry-and-topology-whats-the-difference/

[27] N. Scherich. "Representations of the Braid Groups," 2017, https://vimeo.com/270755890.

[28] Skansens Folkdanslag. https://www.youtube.com/watch?v=fif8Zt1ir70, 2013.

[29] F.A. Zorn. *Grammar of the Art of Dancing*. Dance Horizons. 1905. Pg 227.