Wallpaper Patterns from Looping Strands: The Layer Groups

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Abstract

When creating patterns that are basically flat from 3-dimensional materials, it is tempting to say that they have wallpaper symmetry. A better analysis uses the language of layer groups, which are groups of isometries of 3-space that leave a plane invariant. This article describes the relationship between wallpaper groups and layer groups, providing artistic illustrations of the principles involved. We explain how to create strands with desired symmetry and place them in space to make patterns that display a variety of weaving and looping.

Introduction

Several Bridges authors have connected concepts of wallpaper or frieze symmetry to 3-dimensional objects, recognizing the tension between a theory of 2-dimensional transformations and actual physical objects [9, 11, 13]. Some have mentioned, without necessarily using the term, what the International Union of Crystallographers (IUC) calls the *layer groups* [12]. These are groups of isometries of 3-space, in which all elements preserve a given base plane. This paper has two goals: to provide an accessible exposition of these layer groups and to show new artworks that illustrate the rich variety of patterns available when we liberate the concept of wallpaper from its strictly planar origins.

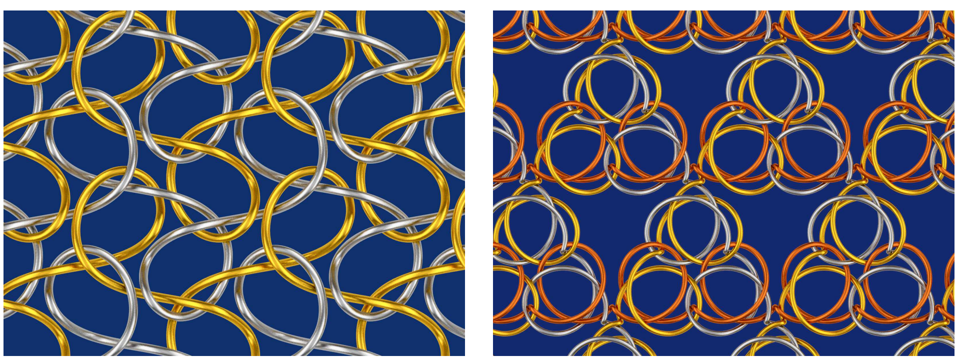


Figure 1: *Two patterns from looping strands. One might say that these have wallpaper symmetry pgg (left) and p3m1 (right), but in each case the set of symmetries of the pattern is a layer group.*

Figure 1 shows two examples of a new series of digital artworks, which grew from my work on mathematical chain mail [6]. Instead of patterns made from linking together mathematically deformed circles, I study patterns made from families of infinite strands. These evoke wallpaper symmetry while also recognizing that variations in over-under weaving of the strands lead to more pattern types than the usual 17. An accurate analysis of the symmetry types of examples like these leads us to 80 pattern types, one for each of the 80 layer groups.

After introducing some preliminary concepts, we begin the investigation of how 17 increases to 80 by showing how one plane group gives rise to four in space. Then we explain the two steps in building patterns from looping strands: First we explain how to place strands with a given symmetry in space to create a pattern, then we backtrack to explain how those strands were created.

The theory of layer groups is closely connected to that of 2-color symmetry in wallpaper. The length of this paper does not permit a catalog of one pattern of each type, but we do offer as a supplement a table to explain how the IUC names for the layer groups connect to the notation of this paper, which is nothing more than a slight expansion of the excellent notation of Coxeter [3], which Conway et al. basically follow in *The Symmetries of Things* [2]. Our goal is not to create any new symbols for these things, but to clarify the situation for anyone who creates or analyzes 3-dimensional flattish things that look as if they have wallpaper symmetry, including Celtic knots, lace, and knitting.

After an overview of the layer groups, I say a word about the artistic creation of the examples. The basic technique involves the Fourier series to which I have returned again and again as a source of flexible periodic shapes. This material is related to knot theory and I hope that others will study the examples of periodic knots and links shown here.

Extending to 3D

Wallpaper patterns are classified by their symmetry groups, which are collections of the Euclidean isometries (translations, rotations, reflections, and glide reflections) whose translations can be generated by two independent ones. (Technically, we should refer to isomorphism classes of these groups, so that, for instance, a large-scale infinite checkerboard and a small-scale infinite checkerboard are said to have the same symmetry group.) We build on the famous result that the symmetry group of every wallpaper pattern, as long as it has translational symmetry in two independent directions, is isomorphic to one of 17 groups, called the *wallpaper groups*. We refer to a general wallpaper group as G and use the IUC notation to refer to particular ones, such as pg, pmg, p3m1, and so on.

How can a wallpaper group, defined as a set of isometries of the Euclidean plane, be construed to act on 3-space? It is productive to consider a particular Euclidean isometry of the plane, say α. Formulas for such isometries are especially convenient in complex notation, so we use $z = x + iy$, where $i^2 = -1$, as a complex variable in the plane and coordinatize points in 3-space as (z, w), where z is a complex number and w, the up-down direction, is real. Since α is a transformation of the plane, we use $\alpha(z)$ to record the place to which the point z is mapped.

If we wish to extend α to an isometry of 3-space, we have few choices: planes parallel to the base plane $w = 0$ have to end up at the same distance from that plane. This means that there are only two ways to extend α: leave the w coordinate alone or flip it to its negative. Let's define the *trivial* extension of α by $\check{\alpha}(z, w) = (\alpha(z), w)$, which says that α just acts on the plane and leave the up-down direction alone. The only other isometry of space that matches α on the base plane is $(z, w) \to (\alpha(z), -w)$, flipping the up-down direction and performing the motion $\alpha(z)$. If we name the reflection through the xy-plane as $\sigma_{xy}(z, w) = (z, -w)$, we can denote this second extension as $\check{\alpha} \circ \sigma_{xy}$, a composition of the trivial extension and the reflection. The reflection and the isometry of the plane always commute, so we need not be careful about the order.

Let us call the extension $\check{\alpha} \circ \sigma_{xy}$ the *reflection extension* of α, though it is not necessarily a reflection of space. For instance, if we start with σ_x, the reflection of the xy-plane across the x-axis, the reflection extension is a 180° rotation about the x-axis, turning the plane over. When it comes to naming the layer groups, a wallpaper group with a symbol m can become a layer group with a 2, when a reflection becomes a half-turn. By contrast, the reflection extension of a translation is not a rigid motion, but rather a glide reflection in the plane spanned by the translation and the w-axis.

Working at the group level, suppose that G is a wallpaper group. If we extend all its elements trivially, we get a group isomorphic to G. Let's call this the *trivial extension* and denote it simply as G, forgiving a small abuse of notation. At the other end of the spectrum, we could form a new group by including all elements of G as well as their reflection extensions. The resulting set of transformations does form a group, which we call the *double extension* and denote as the direct product $G\times\mathbb{Z}_2$, where $\mathbb{Z}_2$ is just the additive group of the integers $\{0, 1\}$ modulo 2, corresponding to the group of transformations $\{\sigma_{xy}, \sigma_{xy}^2 = \iota,$ the identity$\}$. These are the extreme possibilities for *layer group extensions* of G, by which we mean groups of isometries of space whose restriction to the base plane is G.

To understand the possibilities in between, consider what happens if we have both extensions, $\breve{\alpha}$ and $\breve{\alpha} \circ \sigma_{xy}$, in a layer group. Being closed under composition and inverses, the group must include $\breve{\alpha}^{-1}$ and hence $\breve{\alpha}^{-1} \circ \breve{\alpha} \circ \sigma_{xy}$, which is just σ_{xy}. Having this reflection, the layer group includes the trivial *and* reflection extensions of all its elements, and is hence the double extension of the group. The important realization is this:

> If a layer group extension of a wallpaper group is neither the trivial nor the double extension, then for each element α of G, the layer group contains *either* $\breve{\alpha}$ or $\breve{\alpha} \circ \sigma_{xy}$. In other words, the extension to space of every transformation in G either preserves the up-down direction or flips it.

This is all we need to consider some examples, but let us preview how the count proceeds, for each of the 17 wallpaper groups, we know that there must be 17 layer groups that are trivial extensions and 17 more that are double extensions. How shall we count the remaining 46 layer groups?

Example: extensions of pg

The wallpaper group pg is relatively easy to understand. It is generated by two transformations, a horizontal glide reflection and a vertical translation. We name these as $\gamma_x(z) = \bar{z} + 1/2$, where $\bar{z} = \overline{x + iy} = x - iy$, and $\tau_{ia} = z + ia$. Check that $\gamma_x^2(z) = z + 1$, so the group pg contains a horizontal translation of length 1 and a vertical translation of length a. Figure 2a gives a good way to become oriented to these symmetries. The image can be interpreted in two ways: It could represent a pattern invariant under the trivial extension of pg or a pattern invariant under pg$\times\mathbb{Z}_2$, if the bottom halves of all the self-intersecting pipes are mirror images of what we see on the top. (When I made this image, I used pipes with circular cross-section, but I could cut it with a horizontal plane to break the up-down mirror symmetry.)

To look for other possible layer group extensions of pg, consider extending the generators of pg by their reflection extensions. Figure 2b shows a pattern whose symmetries are generated by $\gamma_x \circ \sigma_{xy}$ and τ_{ia}: The reflection extension of the glide exchanges the vertical displacements of the tubes, so that a loop that goes over becomes a loop that goes under. The vertical translation leaves the ups as ups and the downs as downs.

Figure 2c, by contrast, shows a pattern with actual glide symmetry. It's symmetries are generated by γ_x and $\tau_{ia} \circ \sigma_{xy}$: what was a vertical translation now mirrors the strands across the xy-plane.

The last remaining possible layer group extension would be generated by $\gamma_x \circ \sigma_{xy}$ and $\tau_{ia} \circ \sigma_{xy}$. However, when we analyze the symmetries of this pattern, we detect a positive horizontal glide reflection. Can you find it in Figure 2d? This suggests that our would-be new layer group is isomorphic to the second one we constructed, since it's generated by a positive horizontal glide and a reflection extension of the vertical translation.

These examples show the basic process for counting layer group extensions, as well as the possibilities for an overcount. They also give a good opportunity to explain how the layer groups are named.

When a layer group contains some mirror extensions and some trivial extensions, the set of trivial extensions form a subgroup of the layer group. Since the composition of two reflection extensions is always a trivial extension, returning the plane to its upright position, the subgroup has index two in the larger group (a technical way of saying that half the elements flip and half leave the plane right-side up). This means that both the subgroup and the layer group, when restricted to their actions on the plane, are wallpaper groups. The Coxeter notation for the layer groups (other than the trivial and double extensions) consists of a pair of group symbols, where the first is the full layer group and the second is the subgroup of trivial extensions.

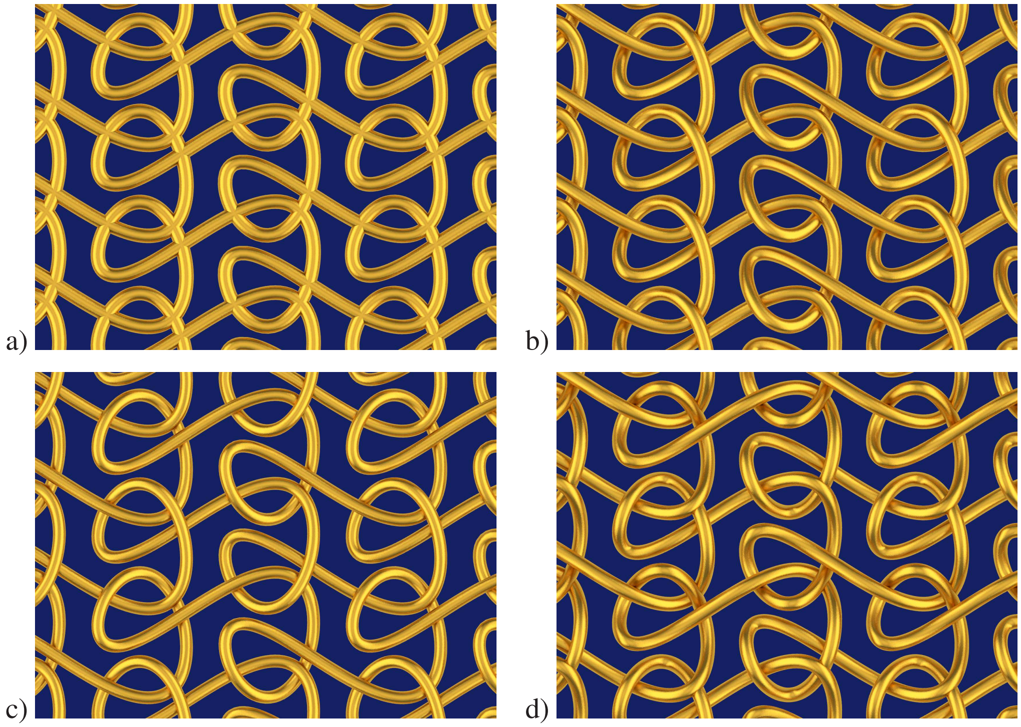


Figure 2: *Patterns invariant under various layer group extensions of the wallpaper group pg.*

Our first example, shown in Figure 2b, has type pg/p1. The only trivial extensions are the translations. By contrast, Figure 2c shows a pattern of type pg/pg. The first pg is generated by γ_x and $\tau_{ia} \circ \sigma_{xy}$, the smaller group of trivial extensions is generated by γ_x and τ_{2ai}.

The third example, Figure 2d, has symmetry type pg/pg, but it has been realized in a way that is different from Figure 2c. The strands in Figure 2c have actual glide symmetry, while those in Figure 2d have half-turn symmetry. This shows that there are multiple ways to realize a given layer symmetry type. It may be interesting to enumerate the possibilities, but we leave that project for another time.

We close this section by connecting the Coxeter notation [3] to that of the IUC [12]. The trivial extension of pg to 3-space is #12 on the IUC list of layer groups, where it is named pb11, a straightforward notation for a glide along a first axis. The double extension is #29, where the name pb$2_1$1. Interpreting b as a glide symmetry is straightforward; the 2 in the second position after the p suggests a rotation through 180°, but the subscript 1 means that the rotation is composed with a translation. This is called a *screw motion*. The layer group that Coxeter called pg/p1, as in Figure 2b, is #9, denoted p$2_1$11; the screw symmetry is prominent in this example: you can turn the band over and slide it along until it matches perfectly. The last situation, Figures 2c and 2d, is #33, denoted pb2_1a, the interpretation of which I leave to the reader to puzzle out.

In summary, there are 4 layer group extensions of pg: the trivial extension, the double extension, and two interesting extensions that led to rich possibilities for our looping strands to weave in and out with one another. Since 4×17 is only 68, some of the wallpaper groups must have more than 4 extensions.

Placing strands in space

Figure 3a shows the source of this investigation: a comforter with a pattern that resembles scalloping tapes that weave in and out to create a pattern that, were it not for those breaks for the up/down weaving, would have symmetry group p4m. Stopping to examine it closely, I realized that techniques from my most recent

Bridges paper [6] could be adapted to model the pattern. Figure 3b shows my model, realized in *Rhino* with *Grasshopper*. In this section, I assume that one knows how to make strands with the scalloping up/down rhythm in the pattern, in order to say more about the role of the symmetry groups in placing the bands.

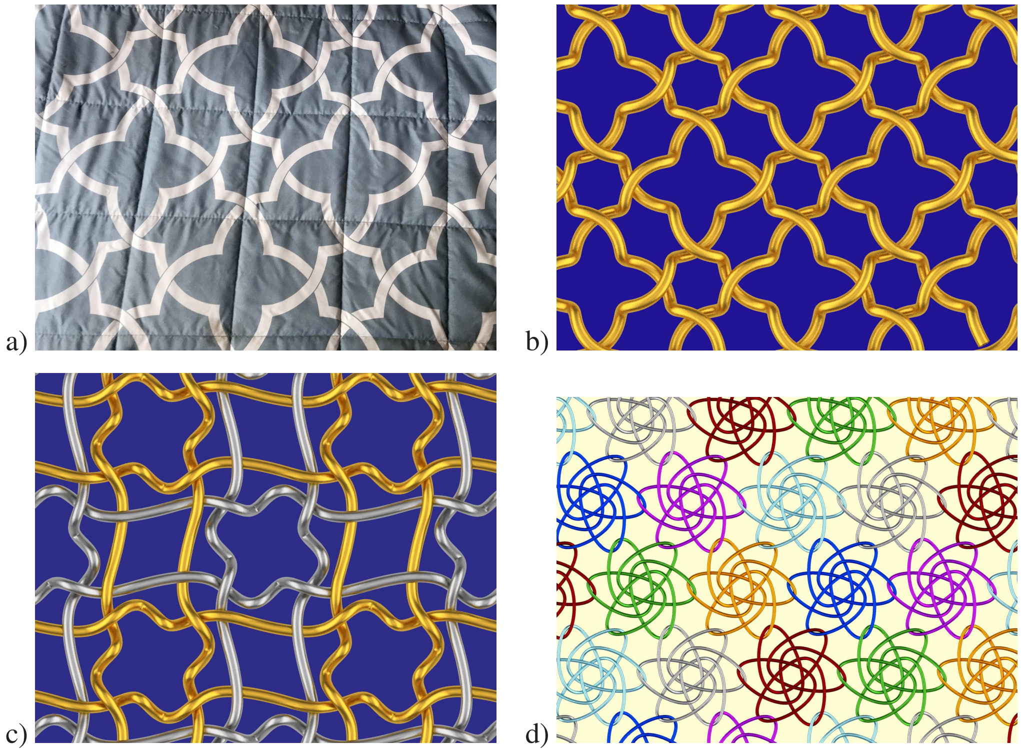


Figure 3: *a) A fabric pattern that sparked this project. b) A mathematical model of the pattern, which has type p4m/p4. c) A pattern of type p4g/p4. d) A pattern of type p6 made from knots instead of strands.*

Focus on a single strand in the woven pattern and orient it along the horizontal. Looking directly down on the strand, which hides any up/down undulation, we would see a frieze pattern of type p1m1: horizontal translational symmetry and mirror symmetry across a vertical axis. Call the vertical mirror reflection σ_y, reflection across the y-axis. Since a turn about that axis takes an over crossing to an under crossing, as we see in the fabric pattern, the actual symmetry of the 3D band must be $\check{\sigma}_y \circ \sigma_{xy}$, not simply $\check{\sigma}_y$.

To create the pattern, move this horizontal band up from the x-axis a bit and introduce 3 more copies of it by repeatedly rotating about the origin by 90°, a transformation we call ρ_4. The symmetry of the strand forces its 180° rotation (ρ_4^2) to be the same physical object as a half-turn of the strand about the x-axis. Translating the configuration repeatedly gives the pattern in Figure 3b, though we turned that figure by 45° relative to the description here, in order to match the fabric pattern.

The pattern type here is #53 in the IUC tables, p422. The Coxeter notation is p4m/p4, indicating that tagging a generating mirror reflection in the wallpaper group for a reflection extension results in a total group of flips/non-flips that is isomorphic to p4m, the pattern type we noticed in the original fabric. To connect with the concepts of the previous section, we note that this layer group is generated by ρ_4 and $\sigma_y \circ \sigma_{xy}$, one of *six* layer group extensions of p4m.

There is a general strategy exemplified in this example, one explained in detail in a recent paper about woven polyhedral symmetry [1]. The idea, as it applies here, is to first to create a strand that is invariant under a subgroup of the desired symmetry group, then place copies of that strand using a selected set of group elements called *coset representatives*. In this case, we found a strand essentially invariant under the frieze group p1m1, though with a reflection extension of the mirror; we might call that group p1m1/p111. It is a subgroup of p4m/p4. All we need to do is spread out translates and rotations of the single strand to make the pattern.

For an easier example, let's walk through the pattern in Figure 3c. This example is simpler because the generating strand has no special symmetry other than periodicity; it just wiggles. Place one strand, say one of the golden ones, horizontally, displaced a bit above the origin. At this stage, the strand is invariant under a group of horizontal translations. Duplicate the strand by vertical translations of the same length as the horizontal period of the band, preserving the invariance under horizontal translations and adding invariance of the configuration under vertical ones. (Theoretically, we assume infinitely many infinitely long bands; in practice, of course, we truncate.) Rotate each set of strands through 90°, 180°, and 270°. This produces the configuration of the golden bands in Figure 3c, a pattern that is invariant under the group p4. The four golden pinwheel shapes near the corners of the image outline the square cell of the pattern. The final step in creating Figure 3c is to create a silver copy of the entire configuration and flip it across a strategically placed mirror axis that connects midpoints of the sides of the square we mentioned. Voilà! Not only does the pattern now have p4g/p4 symmetry, in Coxeter notation, but the gold and silver strands magically interweave perfectly.

This pattern type is #54, $p42_12$, in the IUC tables. All the symmetries of this pattern correspond to direct motions: The mirror symmetries of the 2D projection are all extended to space as half-turns or screw motions. This was also the case in Figure 3b. Although I have no proof at this stage, it seems to me that these groups consisting entirely of direct motions of space offer much greater potential for weaving.

We include Figure 3d here just to say that one can apply the subgroup strategy to the point group of the wallpaper group. In this case, we created a curve invariant under the cyclic group C_6, the point group of p6, and propagated copies of it via translations. The colors are meant to illustrate the 7-color theorem for a torus, which may be the topic of another article.

Creating the strands

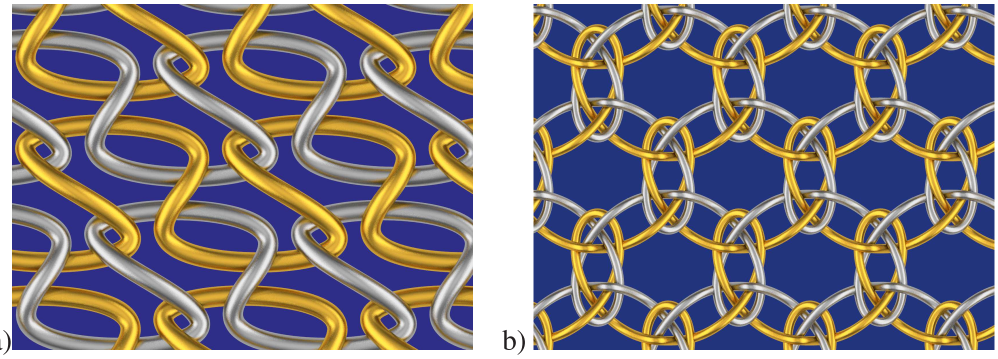
Creating symmetry in plane curves is where I started my adventure with symmetry 25 years ago [7]. Creating strands with given symmetry uses the same mathematics: everything is based on Fourier series. The most general horizontal strand with period 1 can be parametrized, in the complex notation of this article, by

$$f(t) = x(t) + iy(t) = t + \sum_{-\infty}^{\infty} a_n e^{i2\pi nt} = t + \sum_{-\infty}^{\infty} a_n \left(\cos(2\pi nt) + i\sin(2\pi nt)\right),$$

where the coefficients a_n are complex numbers and the frequency coefficients n, are integers. This function satisfies the equation $f(t+1) = f(t) + 1$, the *frieze symmetry condition*. For the cases in this section, it will be simpler to use real notation; for a generic strand with no special symmetry, the complex forms have proved to be more flexible in practice. A very old paper about frieze symmetry, coauthored with Norwegian signal-processing engineer and rope weaver, Nils Kristian Rossing, gives a full account [8].

To create the pattern in Figure 4a, I needed a strand with p211 symmetry. The half-turn symmetry condition, $f(-t) = -f(t)$, leads to the condition that $a_n = -a_{-n}$. When this is translated into real terms, it means that we may only use sine functions in both the x and y coordinates of the constructed curve. For the vertical component of the curve, I wanted to create symmetry under the trivial extension of the half-turn. I used a function of the form

$$w(t) = \sum_{0}^{\infty} b_n \cos(2\pi nt) \text{ to make } w(-t) = w(t).$$



a) b)

Figure 4: *a) A pattern of type p2. b) A pattern of type cmm/p2.*

To experiment with these curves, one needs software that will display the curve and allow the user to wiggle the various coefficients. In practice, I used just a few non-zero terms in the theoretically infinite sum. That was enough to create the pleasantly undulating strand in Figure 4a. I translated the strand at a skew angle, since the group p2 requires no special lattice, and wiggled the parameters until I found something pleasing. I colored the strands silver and gold, not for any mathematical reason, but for consistency of visual style.

Figure 4b requires a strand with mirror symmetry in its plane projection and alternating symmetry in the w direction. This is the same type of strand used for Figure 3b. The recipe is the same, though the appearances are quite different. The mirror condition is easier to implement in real notation than complex: We just use sine functions for the horizontal component and cosine functions for the vertical. Unlike the previous example, here we use sine functions to ensure that $w(-t) = -w(t)$.

For this example, instead of propagating duplicate strands by translation, I offset alternate rows by half a translational unit, creating what appears in the 2D projection to be a glide reflection. Focusing only on the gold strands reveals a pattern of type cm/p1, #10 in the tables, with name c211, due to the extension of the mirror reflection in cm to a half-turn in space. Adding in silver strands by rotating the gold ones about a horizontal axis gives a pattern of type cmm/p2. In the IUC table, this is #22, called c222. This is another example that confirms my experience that beautiful weaving is easiest to achieve for pattern types where the symmetries are direct motions of space, rather than actual mirrors.

Counting wallpaper extensions

Mathematics that I have used before to explain why there are 46 types of 2-color wallpaper patterns [3, 5] is exactly what we need to count the non-trivial layer group extensions. In our examples, every group generator α was extended as either a trivial extension or a reflection extension. For any wallpaper group G, the choice of trivial versus reflection extension for each element can always be described by a group homomorphism

$$\phi : G \to \{0, 1\},$$

where the set $\{0, 1\}$ has the group structure of the integers modulo 2. Given any such homomorphism, we create an extension of G to a layer group by mapping

$$g \to g \circ \sigma_{xy}^{\phi(g)}.$$

Counting the layer groups is exactly the same as counting the number of (equivalence classes of) such homomorphisms. Since the kernal of a homomorphism is always a normal subgroup, this is the same as counting the ways that one wallpaper group can appear as a normal subgroup of another. And there are 46 of those [2, 3, 5]. By the way, the types in Figure 1 are pgg/pg and p3m1/p3.

A word about the art

The images in this article show finished patterns. Along the way, my screen was filled again and again with patterns where the strands ran into one another, stacked on top of one another, and did everything but weave nicely. Achieving pleasant results is highly empirical and individual tastes may vary. I admit that careful examination of some of my figures shows places where the pipes do run into one another just a bit. The non-local nature of Fourier series means that a desirable wiggle in one part of the band might entail an undesirable wiggle elsewhere. This study would not have been possible without software that allowed these experiments. I hope that readers will try their own.

Acknowledgement

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