The Joy of Polar Zonohedra

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Abstract

Polar zonohedra are a special class of polyhedra that have inspired sculpture and constructions in many media. The shape is akin to a faceted version of a U.S. football, with two pointed poles. They can vary in complexity and can range in elongation from a pancake shape to a cigar. Several constructions are presented here, including a little library and a dome large enough to hold several people. Some mathematical basics, historical remarks, applications, fabrication techniques, and variations on the ideas are presented.

Introduction and Structural Properties

Figure 1 shows a "little free library" I built based on a 10-fold polar zonohedron (PZ). A little free library is a weather-tight container, typically installed in one's front yard, where anyone can freely take books or leave books for others. This one is constructed of wooden rhombi of various shapes, arranged with 10-fold symmetry about a vertical axis. In order to construct such forms, one must know the angles for cutting the rhombic pieces and the proper dihedral angles for joining them along their edges. (See Appendix.) The principles apply on scales from small models to architectural structures.

Figure 2 shows a variety of n-fold PZ and illustrates the two parameters one can vary when choosing a form from this family: the order, n, of rotational symmetry and the pitch angle of the edges, θ. With only two degrees of freedom, the range of forms is highly constrained, so PZ have great internal coherence, contributing to their aesthetic appeal. Each consists entirely of planar rhombi. All edges are the same length. There are two special n-fold vertices, called "poles," where n rhombi meet. The entire polyhedron has central symmetry, e.g., every face has an equal opposite parallel face. The line connecting the poles is an axis of n-fold rotational symmetry. The vertices are all of degree 3, 4, or n, with the two poles of degree n, the $2n$ vertices adjacent to the poles of degree 3, and all remaining vertices of degree 4.

Figure 1: *Little Zonohedral Library*

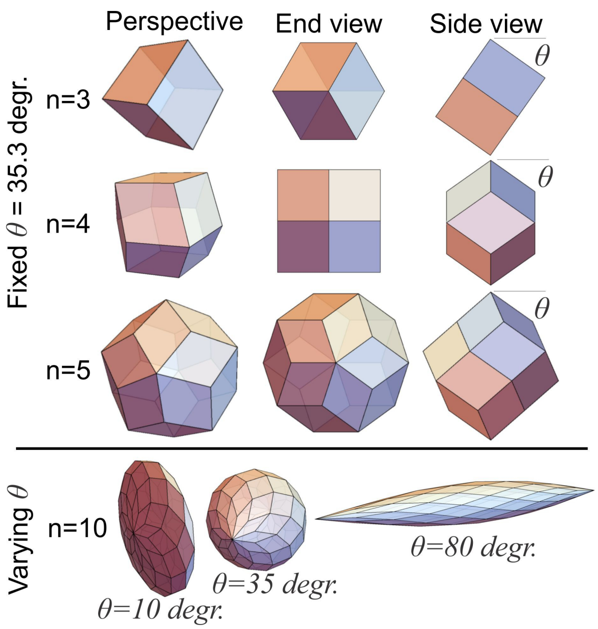

Figure 2: *A Variety of Polar Zonohedra*

Two special cases are: (a) any rhombohedron (a polyhedron built from six identical rhombi, e.g., the special 90 degree case of a cube) is a 3-fold PZ and (b) Kepler's rhombic dodecahedron is a 4-fold PZ. For $n \geq 5$, the two poles are salient for their higher order and it is easy to understand the form as akin to a U.S. football discretized into planar facets. For visualization, it is natural to orient a PZ with its n-fold rotational axis vertical, as the Z axis in space, with the lower pole at the origin. (This is an example of a general preference for orienting objects with a unique axis vertically, because mental space is not isotropic but has its vertical direction distinguished by gravity [9].)

The n edges that meet at either pole are positioned as the "ribs" of a partially open umbrella. The angle θ can be visualized as the degree of closure of the umbrella, measured from the horizontal plane. One can algorithmically generate the complete PZ from the n ribs by filling in the open "V"s between the ribs with rhombi, to create the first "level" of n faces, then repeating the process to make each additional level until the process closes satisfyingly at the opposite pole (which is the vector sum of all n umbrella ribs). See Figure 3 and the animation [10]. It is particularly pedagogical to build one yourself in three dimensions with Zometool [13].

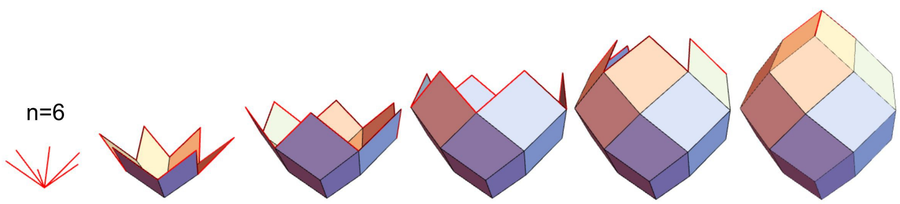

Figure 3: *Filling in "V"s, starting from ribs, to build five levels of faces.*

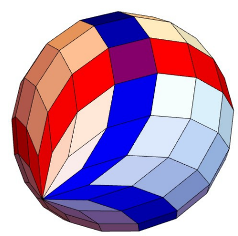

Figure 4: *Two Zones*

This V-filling process is rigorous, but a PZ can also be concisely defined (for a mathematical audience, at least) as the Minkowski sum [19] of n segments that are arranged with n-fold rotational symmetry. Thus they are a rotationally symmetric subset of the more general class of arbitrary zonohedra (the polyhedra which are the Minkowski sum of any arbitrary set of segments) [6].

Rhombi are parallelograms, so each new edge that is created when one fills in the "V"s (per Figure 3) will be parallel to one of the ribs and there are exactly n edge directions. A "zone" of faces is defined as the set of faces which share one edge direction. Two zones are highlighted in Figure 4. One can imagine starting on any face, walking along a zone, and completing the cycle at the starting face. Each face is a crossing of two zones. Zones encircle the polyhedron (like a zig-zagging ribbon equator), so each pair of zones crosses twice, at two opposite parallel congruent faces of the PZ.

With the above observations, it is easy to count the polyhedral components of an n-fold PZ, i.e., the number of faces (F), edges (E), and vertices (V). There are n-choose-2, i.e., $n(n-1)/2$, ways to pick two different starting ribs and each choice determines two zones and thereby two opposite faces, so there are $F=n(n-1)$ faces. Because all the faces are 4-sided, there must be twice as many edges as faces. (Imagine F individual paper rhombi, having $4F$ polygon edges, being taped together in pairs to make half as many, i.e., $2F$, polyhedra edges.) So $E=2F=2n(n-1)$ edges. Then from Euler's formula ($V+F=E+2$) we can calculate that there are $V=n(n-1)+2$ vertices. (It is also insightful to visualize directly how $V=F+2$ in any PZ, by associating each face with its most clockwise vertex, spinning around the axis in a consistent direction, thereby making a bijection between the faces and all the vertices except the two poles.)

The $n(n-1)$ faces of a PZ can be grouped as n congruent rhombi at each of the $n-1$ levels. The "top" angle of each rhombus, i.e., the angle closest to a pole, grows as you move away from the pole, reaching a maximum at the level closest to the "equator," then shrinks again. By symmetry, the angle for the level i from the top is the same as for level i from the bottom, so there are (at most) Floor[n/2] different face shapes. If n is odd, there are an even number of levels and each shape appears in two different levels, but if n is even, the middle level is unique. (By a method discussed below, the number of different face shapes can be reduced by one from this upper bound.)

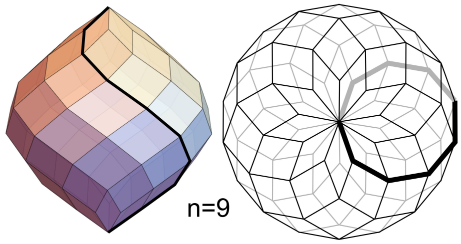

Figure 5: *Surface Helix*

A lovely feature of any PZ is the spiral path of edges I call a "surface helix." Start at a pole, choose any of the *n* rib edges and mentally follow it to a 3-way vertex where you can choose to take either the left or right fork, then continue following edges "straight across the intersection" at each 4-way vertex, until your path leads you to the opposite pole. One surface helix is highlighted in Figure 5. The edge directions follow the cyclic ordering of the ribs, e.g., ribs i, i+1, i+2, ..., mod n. The rotational symmetry of the umbrella ribs implies a consistent change in angle from one to the next and every edge has the same slope relative to horizontal. Therefore this path of *n* edges going from pole to pole takes the form of a discrete helix, i.e., *n* equal chords covering one revolution of a circular helix. Figure 5 also shows that the projection of a surface helix onto the *XY* plane is a regular *n*-gon. The helix diameter is half of the PZ's widest diameter. Every PZ has *n* right-handed surface helices and *n* left-handed ones. For large values of *n*, the segments are short relative to the diameter, so the discrete helix approaches a true continuous helix.

History and Architectural Applications

The idea of PZ was apparently discovered twice independently. The Russian crystallographer E.S. Federov first described zonohedra and PZ in 1885 in the context of minerology, but his book was unknown for a long time in the West [15]. The first record in a Western language of the general PZ class or examples that are 5-fold or higher is a 1937 *Mathematical Gazette* article by C.H.H. Franklin. He presents PZ in the dubious context of a paper titled "Hypersolid Concepts and the Completeness of Things and Phenomena" [7]. Oddly, he claimed that with their aid "it should become easy to picture and think directly in the space-time continuum and so proceed towards consciousness of the Completeness of Things." I have not experienced that, but despite this mysticism, the paper contains the sound idea of projecting from *n*-dimensional space to make 3D models, in particular how using "the ribs of an umbrella at any chosen angle to the polar axis" as projection directions produces PZ as shadows of *n*-dimensional hypercubes. Franklin reports that large drawings and physical models were displayed at a 1937 British Association mathematics meeting, but sadly his paper doesn't include any photographs or diagrams.

Little about PZ has been written for a general audience. As of this writing (early 2021), Wikipedia doesn't mention polar zonohedra. The classic 1895 *Mathematical Recreations and Essays* by W.W. Rouse Ball was expanded and revised in the mid-1900s by H.S.M. Coxeter, who added a sumptuous Polyhedra chapter with three terse sentences on PZ [3]. Coxeter describes PZ and their role as 3D projections of hypercubes more deeply in his 1947 university-level text *Regular Polytopes* [6]. A 1963 *American Mathematical Monthly* paper by Chilton and Coxeter analyzes the limiting shape for large *n* [5].

The inventor and designer Steve Baer learned about zonohedra and PZ from this math literature. He began making alternative architecture based on zonohedra in the 1960s, including buildings with a PZ shape. His 1970 *Zome Primer* shows a 7-fold PZ [2]. Baer designed a system that allowed easy construction of accurate physical models [14]. Starting 1971 ZomeToy was produced in limited quantities before Marc Pelletier and Paul Hildebrandt redesigned it as a second-generation Zometool that became available in 1992. This ignited wider interest in zonohedra, including PZ, and certainly was my spark.

Baer promoted the use of zonohedra as building units that allow for a wide variety of forms with a small inventory of reusable components. The PZ dome is one particularly symmetric form in this family. The earliest architectural-scale PZ I have found is Baer's central 8-fold dome at the Lama Foundation, near Taos, NM, circa 1970. In the intervening decades others gradually came to follow this path and an internet search for "zome" turns up a variety of recent examples of PZ homes, greenhouses, sheds, yurts, etc.

Modern computer graphics software easily allows visualization of geometric structures. Russell Towle wrote software that generates PZ images, described in an article in the Graphics Gallery section of the *Mathematica Journal* in 1996 [17].

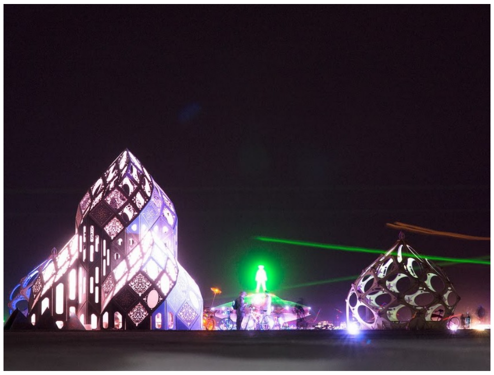

Fig. 6: *Bell's "Quasicrystalline Conjunction"*

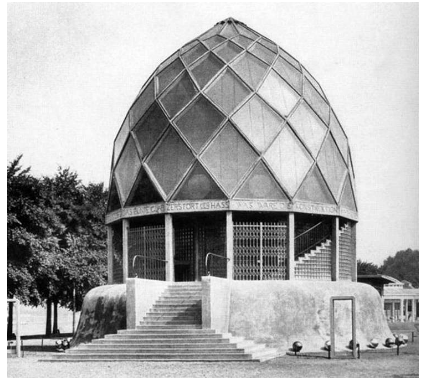

Figure 7: *Taut's "Glass Pavilion"*

Fig. 8: *Foster's "Gherkin"*

A few math-inspired, artistically aware, hands-on designers and builders have embraced PZ. The works of Baer, Pelletier, Hildebrandt, Chris Palmer, and Towle have all impacted me [14], but a most gifted practitioner was certainly Rob Bell. For a decade, he designed, fabricated, and organized groups of builders to construct monumental PZ structures as temporary art-temples in the Black Rock Desert at the annual Burning Man festivals [4]. He also wrote a plug-in to the popular 3D design package SketchUp to create zonohedral structures. Figure 6 is Bell's "Quasicrystalline Conjunction" at Burning Man 2011.

To fully appreciate PZ lengths and angles, it is instructive to look at Bruno Taut's 1914 Glass Pavilion (Figure 7), built as the centerpiece for the first Deutscher Werkbund exhibition [16]. While renowned as an influential example of German Expressionism in architecture and groundbreaking in its use of glass both for structure and for visual effect, I find it to be a failure in terms of geometric aesthetics. Anyone familiar with the beautiful surface helices of a PZ must be viscerally disturbed by the lines of the pavilion's glass dome. It is so close to a 14-fold PZ, and yet so far. The lack of mathematical underpinnings resulted in meandering lines that shout "missed opportunity!" Of course, this was decades before PZ were popularized, so one can not fault Taut for not availing himself of the mathematics.

Another famous example that is not quite a PZ is London's "Gherkin" skyscraper (30 St. Mary Axe) by Norman Foster and Partners. See Figure 8. It is proportioned slightly differently than a true 18-fold PZ and rounded at the top, but the helices are sensitively drawn and the panel coloring nicely emphasizes spiral zones. The designers chose to highlight left-handed screws along the surface, which raises the interesting question of how architects should choose the chirality of helical structures such as spiral staircases or PZ zones. For thoughts on this question and some (mostly left-handed) examples, see [1].

A Few Constructions

An online image search for "polar zonohedron" will find many ways that various designers have used them as the basis for physical objects from paper, cardboard, wood, 3D-printed plastic, metal, or stained glass. Some have solid rhombic faces while others physicalize only the edges. Some are functional lamps, hats, or shelters, while others are purely aesthetic. Anyone can build paper models with scissors and tape if they have a template for the faces. To design your own example, choose the order, n, and pitch angle, θ, then calculate the face angles of the rhombi as in the Appendix. You do not need to know the dihedral angles between the face planes when making a paper model. If you tape the edges together properly, the tape acts as a hinge and automatically finds the correct dihedral angles as the model closes up. This is true when building any convex polyhedron from rigid plates, by Cauchy's theorem.

Cardboard is thin enough that it can be assembled similarly to paper, without having to plan for the dihedral angles. Figure 9 shows a 10-fold PZ dome (containing the author) assembled from laser-cut cardboard rhombi. Each edge was cut with a dotted line for a flap that is folded inward. Flaps from adjacent faces are paired and held together with small black binder clips on the inside. An elliptical opening in each rhombus makes for a light structure with an open feeling. It is truncated at the point that includes just six and a half levels of faces (rather than the nine levels in a full 10-fold PZ).

Figure 9: *Cardboard Dome*

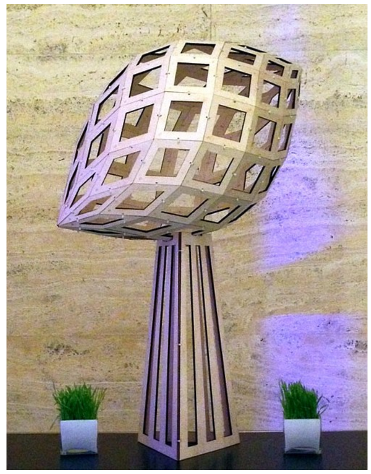

Figure 10: *Superbowl Activity*

Figure 11: *3D-Printed "Zonodome"*

When making a polyhedron from wood, the material thickness and rigidity necessitate that one take into account not just the face angles but also the dihedral angles. Figure 10 shows a wooden "football" in the form of a 10-fold PZ. The edges of the rhombi were beveled so they butt and can join cleanly with cable ties. To view it from various angles and get a closer view of the edge connections, a video can be seen online [12]. I designed it as a hands-on math/art activity for the youth at a large Superbowl party, to engage them while the adults watched the game. (Note: I hate football, but the commission paid well...) I set the pitch angle to match the proportions of a U.S. football and chose n=10 so that the total number of rhombi (90) would take a reasonable amount of time to build. Tips for group-assembling laser-cut wood sculpture with cable ties are given in [8].

The little library of Figure 1 is another example of a wood PZ. The fabrication process is shown in the video [10]. The parts are small, so I used a laser-cutter, but one can also cut very accurate wooden rhombi with a table saw. When making wood polyhedra, I often bevel the panel edges to half the dihedral angle for a flush joint. This seems reasonable if you think of each edge in isolation and will work fine for making wooden Platonic solids. But it doesn't always work with arbitrary polyhedra because some panels might still jam at the vertices. For perfect joinery, there exists a ray inward from each vertex, such that the beveling plane at each edge is the plane that includes the edge and this ray. For rotationally symmetric solid angles, the axis can be used, as it bisects all the dihedrals, but in general, bisecting any two dihedrals determines a ray that might not bisect the remaining dihedrals. When small corner jams happen in practice, I typically sand off a little wedge from the inside of a panel to eliminate the interference.

Figure 11 shows a 9-foot tall, 16-fold PZ "Zonodome" large enough to hold several people. I led dozens of volunteers in its assembly at the 2018 Construct 3D Conference. The vertices are 3D-printed plastic connectors and the struts are wood dowels. The connectors are slightly different for each level, with sockets for the dowels aimed in the appropriate directions. A full description of the project, including photos of the event, assembly instructions, STL files for 3D printing the connectors, and software to create custom connectors for domes with your choice of n and θ, is available online [11].

Design Choices

In any design task, one considers both aesthetics and functional issues, among other factors. In designing a PZ form, mathematics may provide some guidance for choosing n and θ. For example, a larger value for n gives a rounder effect, but there will be more parts to deal with and pointier faces near the poles. The face angle of the rhombi meeting at the poles is at most $360/n$ for pancake-like θ values and gets sharper for cigars. Thin, sharp components can pose difficulties for assembly. For large n, designers may modify the PZ structure near the poles, eliminating edges to reduce the sharpness and density of struts. E.g., online photos show that some beams terminate a bit before the top of the Gherkin.

For a fixed n and edge length, one can optimize θ for volume. The volume of a pancake or a cigar is zero; in between, there exists a pitch of maximum volume. It turns out that this optimum occurs when θ = ArcTan[1/Sqrt[2]], approximately 35.26 degrees, for any n. (Mathematicians may note that this particular value makes the set of ribs a "eutactic star," i.e., an orthogonal projection of a right-angled n-dimensional coordinate basis [6].) This pitch angle gives the cube when n=3 and generally makes the rhombi at levels n/3 and 2n/3 be squares if n is a multiple of 3. For n=4, this θ gives the classic rhombic dodecahedron. (One can derive this value for θ knowing, from the Appendix, that the height is proportional to Sin[θ] and the radius is proportional to Cos[θ], so the volume is proportional to $\text{Cos}^2[\theta]\text{Sin}[\theta]$. Differentiating, setting to zero, and solving gives the optimum θ. The same argument finds the same volume-maximizing angle of 35.26 degrees for a cone or any n-sided pyramid of fixed slant-height.)

Analogously, in some applications one might seek the value of θ that maximizes the PZ surface area, but I know no of analytic optimization result for this. Tabulation shows that the optimum depends on n. If n=3, the surface area is also maximized at 35.26, again giving the cube, but as n increases, the area-optimal θ decreases slightly to around 33.5 degrees for large n.

If one wants rhombi of a particular shape at a specified level, one can choose the pitch angle to make that face angle equal to a desired value or its supplement. (A rhombus with a given vertex angle is congruent to one with the supplementary angle.) E.g., one pitch angle that is sometimes a natural choice is θ=45 degrees. For this value, if n is even, there is a center level of square faces. It also makes the limiting shape (for large n) approach the surface of revolution of a natural sinusoid with slope 1 at the poles [5].

One can also choose special values of θ to reduce the inventory of parts. Even though the "top" angle of a PZ's faces always increases towards the equator, we can choose θ to have two top angles be supplementary, making the rhombi at two different levels be the same. A nice example happens with a 5-fold PZ. By symmetry, for any θ the faces will be congruent at levels 1 and 4 and also at levels 2 and 3, so there are generally two shapes of rhombus. But by writing an equation setting those two face angles to be supplementary, one can solve for the special value of θ that makes all twenty faces congruent. The result is Federov's rhombic icosahedron—a 5-fold PZ in which the faces are all golden rhombi [15]. Generally, for larger n one can pick which two pairs of levels to make congruent in this way.

Keeping θ constant, it follows from the geometry of the umbrella ribs that if k divides n, all the face angles of a n/k-fold PZ also appear in an n-fold PZ. These repetitions may be useful if one has a construction system which allows disassembling a PZ into its faces and reassembling them into a new structure. Rob Bell described such a system and the reuse of physical components in his blog [4].

Variations

PZ lie at the core of a rich space of design possibilities. There isn't room here for a full discussion, but a few ideas can be sketched. See Figure 12. Each zone of a PZ (or any zonohedron) may be either stretched or compressed by changing the lengths of all the edges that share a common direction. In the limit, any zones can be completely removed by shrinking their length to zero. One can also choose to add new zones in new directions, thereby creating a general zonohedron. (This can be formalized either as taking its Minkowski sum with a line segment or as taking the convex hull of it and a translated copy of itself.) One special case of this is worth noting: if one starts with a PZ and adds a new zone of edges in the direction of the n-fold axis, the process preserves the PZ's n-fold rotational symmetry. Although this axial zone disrupts the surface helices, it can look good and be useful for making a taller structure (especially when n is odd and only the upper half of the surface is being constructed). In the case of Federov's (5-fold) rhombic icosahedron, this produces Kepler's rhombic triacontahedron, comprising thirty congruent rhombi with six edge directions.

PZs can be used as building blocks in larger structures. For example two identical PZs can be fused by translating a pole of one to a vertex of the other. Edges will coincide along the "seam" because the surface helices are congruent. Assembling multiple domes in this way leads to natural architectural clusters. In unpublished work, Pelletier developed some lovely examples and models.

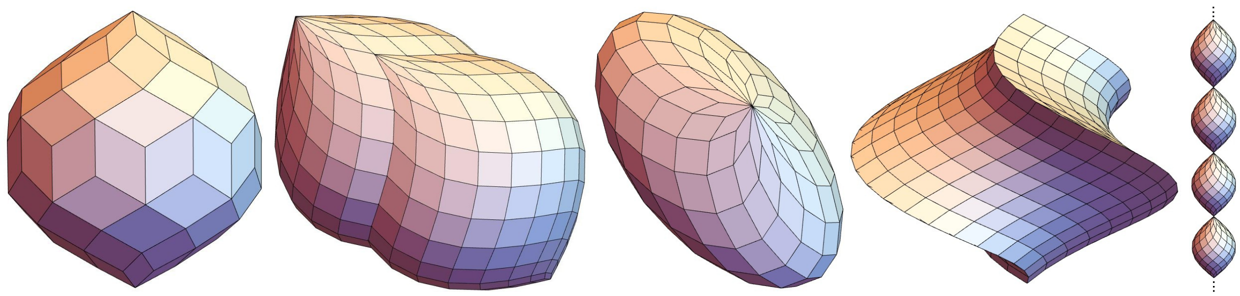

Figure 12: *Variations: Extra axial zone; Two fused PZ; Elliptical cross section; Spirallohedron; and Complete PZ.*

The most impressive example I have seen with PZ as components in a larger design is the 5-meter Zometool structure constructed at Bridges 2009 as a memorial to Chris Kling [20]. It has the form of an edge-model of the rhombic triacontahedron with approximate 10-fold PZ as the struts. (Zometool doesn't have the angles needed for an exact 10-fold PZ, but allows for close approximations [13].) The group photo [20] shows Chris Palmer and Paul Hildebrandt wearing PZ hats that Palmer created.

One lovely PZ generalization is to stretch the umbrella ribs from a circle to an ellipse by scaling all X coordinates. The rhombi transform to parallelograms, but this is a linear mapping, so it preserves parallelism between edges and maintains planarity of faces. The necessary dimensions can be worked out straightforwardly from the vertex coordinates. There is now a third design parameter, and one can cover elliptical footprints. While trivial in software, constructing physical structures of this sort requires care to keep track of the many different edge lengths, face angles, and dihedral angles [4].

Another beautiful PZ variation is the "spirallohedron" [18]. These elegant polyhedra are related to PZs, but are nonconvex and of lower rotational symmetry. Around 2002, Russell Towle discovered them and how some can pack space. His method began with the fact that any n-fold PZ can be dissected into n-choose-3 parallepipeds (one for each way of choosing three edge directions) [6]. By carefully removing certain outer cells from a dissected PZ, he arrived at the spirallohedron forms.

Zonohedral surfaces that are infinite and/or nonconvex can be constructed from parallelograms [10]. E.g., the finite PZ discussed in this paper can be viewed from a broader perspective as just one lobe of "the complete polar zonohedron"—a chain of lobes extending infinitely in both directions, like the surface of revolution of an infinite sine wave. This is akin to how the "dunce cap" notion of a "cone" can be viewed as just a finite portion of the more mathematically natural "complete double cone."

Conclusion

Studying polar zonohedra gives insight into a rich family of geometric forms that can be customized, generalized, and applied in many creative ways. Unlike Fuller's geodesic domes (in which all the vertices lie on a spherical surface) PZ structures inherit the parallelism, modularity, and expandability of general zonohedra. Their structural logic translates into wide-ranging applicability and strong visual coherence.

This paper is dedicated to four inspired PZ explorers we have recently lost: Russell Towle, Chris Kling, Marc Pelletier, and Rob Bell.

Appendix. Geometric Details: Angles, Size, and Coordinates

Using trigonometry and basic vector operations it is straightforward to find formulas for face angles, dihedral angles, vertex coordinates, pole-to-pole height, and maximum diameter of an n-fold PZ of pitch angle θ. We define the pitch of the edges to be measured relative to horizontal, so setting θ near zero makes a pancake, while θ near 90 degrees produces a cigar. To contain a positive volume, let $0 < \theta < 90$ and $n \geq 3$. We give coordinates and dimensions based on edges of length one, which can be scaled as required if one seeks a target diameter or height. We write $\boldsymbol{g}_i$ to refer to the individual umbrella rib vectors (thinking "g" for "generator"). With the lower pole at the origin, the (X, Y, Z) coordinates for the i^{th} of n unit-length ribs equally spaced around the Z axis is:

$$\boldsymbol{g}_i = (\text{Cos}[\theta]\ \text{Cos}[360\ i/n],\ \ \text{Cos}[\theta]\ \text{Sin}[360\ i/n],\ \ \text{Sin}[\theta])$$

Because the dot product (•) of two unit-length vectors is the cosine of the angle between them, we can calculate the face angle at the i^{th} level as

$$f_i = \text{ArcCos}[\, \boldsymbol{g}_0 \bullet \boldsymbol{g}_i \,] = \text{ArcCos}[\, \text{Cos}[360\ i/n]\ \text{Cos}^2[\theta] + \text{Sin}^2[\theta]\ ]$$

It is easy to express the dihedral angles via the vector cross product (×), which gives a vector orthogonal to the plane of its two arguments. The dihedral is given as the angle between the face normals, so zero indicates parallel faces. A helper function to normalize a (non-zero) vector to unit length is $\text{N}[\,\mathbf{v}] = \mathbf{v} / |\mathbf{v}|$. Note (by tracing the zones to see where they intersect) that the pattern of indices of which ribs to use to specify adjacent faces is different for the level 1 dihedrals than for the general level i when $i > 1$:

$$dihedral_1 = \text{ArcCos}[\ \text{N}[\boldsymbol{g}_1 \times \boldsymbol{g}_2] \bullet \text{N}[\boldsymbol{g}_2 \times \boldsymbol{g}_3]\ ]$$

$$dihedral_i = \text{ArcCos}[\ \text{N}[\boldsymbol{g}_1 \times \boldsymbol{g}_i] \bullet \text{N}[\boldsymbol{g}_1 \times \boldsymbol{g}_{i+1}]\ ] \quad (\text{for } 1 < i < n)$$

The vertical projection of any edge is $\text{Sin}[\theta]$, so the total pole-to-pole distance is:

$$height = n\ \text{Sin}[\theta]$$

The maximum horizontal radius can be worked out via similar triangles in Figure 5.

$$maximum\ radius = \text{Cos}[\theta]\ /\ \text{Sin}\ [180/n]$$

Coordinates of the individual vertices are needed if making a computer rendering or a 3D-printed model. They can also be used to answer specific dimensional questions like "What would the base diameter be if truncated at level k?" Keeping in mind the parallelogram image of vector sums, it is easy to see that each vertex is a sum of generators that have contiguous indices, e.g., one of the vertices is $\boldsymbol{g}_3+\boldsymbol{g}_4+\boldsymbol{g}_5+\boldsymbol{g}_6$. So it is natural to index vertices by the starting and ending indices in this sum, e.g., this vertex is called $\mathbf{v}_{3,6}$. Then $\mathbf{v}_{i,i} = \boldsymbol{g}_i$ is the i^{th} generator and by convention an empty sum gives the identity element, i.e., $\mathbf{v}_{i,j} = (0,0,0)$ when $i > j$. So the general vertex is:

$$\mathbf{v}_{i,j} = \sum_{k=i}^{j} \boldsymbol{g}_k$$

With this indexing system, a typical face (the one with its uppermost vertex at $\mathbf{v}_{i,j}$) is the polygon with vertices $\mathbf{v}_{i,j}$, $\mathbf{v}_{i+1,j}$, $\mathbf{v}_{i+1,j-1}$, and $\mathbf{v}_{i,j-1}$. These formulas can be evaluated with a calculator or coded in any programming language. I have provided a reference implementation in Mathematica in the supplement.

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