

# Using Archimedean Spirals to Explore Fractions

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## Abstract

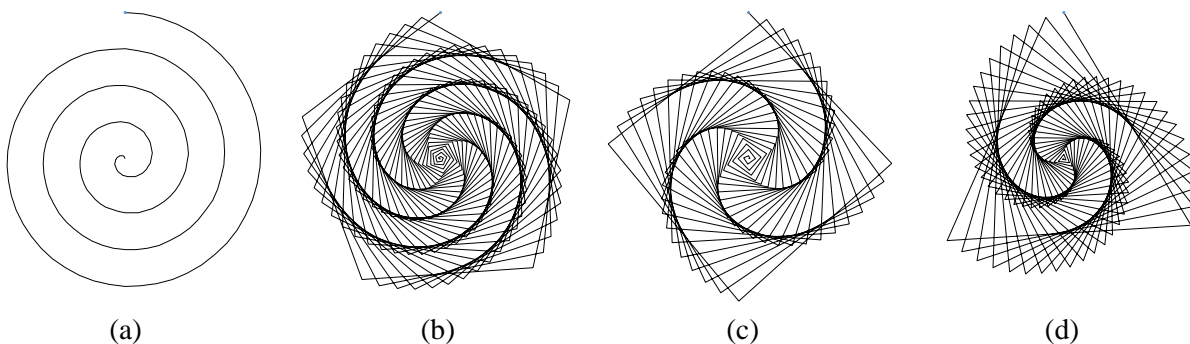
This workshop introduces users to the *Spirals* Excel file which was developed to encourage exploration of the relation between geometric and numeric patterns for young learners. This can create a firmer understanding of multiplication, division, and fractions even before these concepts are formally introduced in a highly interactive and visual fashion. Such explorations may feel more like creating art than learning math. Although developed for independent exploration, the *Spirals* file could readily be employed in classroom settings for students with Excel-enabled tablets, or via smart-boards to amplify a variety of topics in elementary and early secondary education.

## Introduction and Motivation

Mathematics is the study of patterns, some of which are numeric and some of which are geometric. Learning is enhanced if one is able to see a connection between these two ways of knowing. This is particularly true for those who are transitioning from play to more formal learning situations. Research suggests that technology can enhance learning in the early elementary years, especially if learning geometric shapes [6]. When the images created are artistic in nature, young learners may view this exploration as play.

One could argue that purposeful play creates situations where the individual becomes primed to learn from that play. This is consistent with Stein's view of teaching mathematics based on *The Triex: Explore, Extract, Explain* [5]. He argues that we should encourage our students to explore and gather data, seek to extract some sort of pattern from those explorations and ultimately seek to explain what has been found. He notes that even if an individual is unable to achieve the level of "Explain," they have been primed to appreciate the explanation provided by an instructor or another student [5]. Such play encourages learning and deepens understanding and can certainly occur prior to formally learning the concepts of multiplication and division in the middle elementary years [3].

The *Spirals* Excel file allows users to create polygons, stars, and spirals via three click-controlled parameters. The file begins by using the strategy proposed in [1] of teaching whole number patterns in the early elementary grades using regular polygons and stars based on two parameters:  $n$ , the number of vertices (and sides) in a regular polygon; and  $J$ , the number of jumps between vertices. The file adds one additional whole number parameter, the radius reduction factor,  $r$ . Figure 1 shows a bit of the scope of images one can create using *Spirals*. These images range from traditional spirals such as 1(a) to the swirling "twisted polygons" shown in 1(b) – 1(d). In the following sections we will lay out how workshop participants will explore creating images such as these, as well as more complex visual images, using just three parameters.



**Figure 1:** Sample images created with the *Spirals* Excel file.

## Exploring the Parameters Controlling the *Spirals* Excel file

The workshop's initial task is to get participants comfortable using the file. Up and down arrows  $\blacktriangle$  control the three parameters. These scroll arrows allow users the ability to rapidly adjust values (and quickly test hypotheses, even if those hypotheses are informal in nature) and it provides younger users greater flexibility in using the file. Figure 2 shows the dashboard with values set to the values used to create Figure 1(a). The parameter ranges are:  $1 \leq J \leq 50$ ;  $0 \leq r \leq 300$ ; and  $3 \leq n \leq 51$ . Three click-boxes allow the user to alter the image (by showing parent polygon red vertex dots (see Figure 3), by showing a blue vertical radius, and by adding (J, r, n) values in the upper left corner of the image). The first two provide useful teaching anchors and the last is useful for those wanting to transfer images to Word with parameter information included. A fourth click-box allows you to show or hide key ratios created by the three parameters. These ratios, two of which are shown below, help in identifying patterns in the images. For example, there are 4 coils in the spiral in Figure 1(a). These ratios can also provide important teaching tools for use in a classroom setting.

The dashboard consists of three yellow control boxes with scroll arrows and a table of ratios.

- Box 1: **J vertex Jumps between points** (value: 1)
- Box 2: **r 200 1/r reduction factor per jump (for r > 0)** (value: 200)
- Box 3: **r = 0 for stars and polygons n vertices in parent polygon** (value: 50)

Below the boxes is a table of ratios:

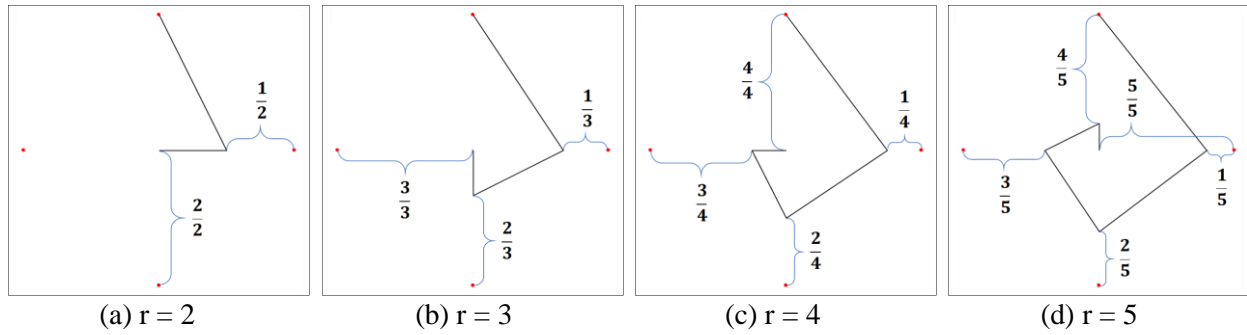
	decimal	fraction
<input checked="" type="checkbox"/> <b>Show spiral key ratios</b>		
<input type="checkbox"/> <b>Show parent polygon vertices</b> The type of spiral is tied to the ratio n/J	50	50
<input type="checkbox"/> <b>Show vertical radius</b> Number of times around the circle is m/(n/J)	4	
<input type="checkbox"/> <b>Show (J, r, n) labels</b>		

**Figure 2:** *Spirals.xlsx* dashboard showing values used to create Figure 1(a) with image and ratio options.

When  $r = 0$  you create polygons and stars. If  $J = 1$  and  $r = 0$ , the resulting image is a regular polygon with  $n$  vertices and  $n$  sides and a top peak vertex (the peak vertex is the initial vertex and jumps are counted clockwise in the *Spirals* program). When  $J > 1$  and  $r = 0$ , the resulting figure depends on whether  $J$  and  $n$  have common factors. Let  $\text{GCD}(n, J)$  be the greatest common divisor function. If  $\text{GCD}(n, J) < J$ , the result is an  $(n/\text{GCD}(n, J))$  point continuously drawn star. Continuously drawn stars require that you connect from vertex to vertex *without* lifting your pencil until you complete the cycle and return to the initial vertex [4]. A five-point star, where you start at one vertex and connect every other vertex, is the most commonly drawn star ( $n = 5$  and  $J = 2$ ). But many other stars can be drawn if  $n > 6$ . An interesting exercise to ponder is why is it impossible to create a six-point continuously drawn star? If  $J$  is a divisor of  $n$ , the resulting image is an  $n/J$  sided regular polygon. Each of the above rules can be extracted by playing with the parameters. In the process, the person doing the exploring will solidify their understanding of multiplication, division, and common divisors, even if they have an incomplete understanding of those concepts. A fun exercise is to find out how many types of stars you can make with vertices from 11, 12, and 13 sided regular polygons (or 23 and 24). This can lead to a nice discussion of prime and relatively prime numbers.

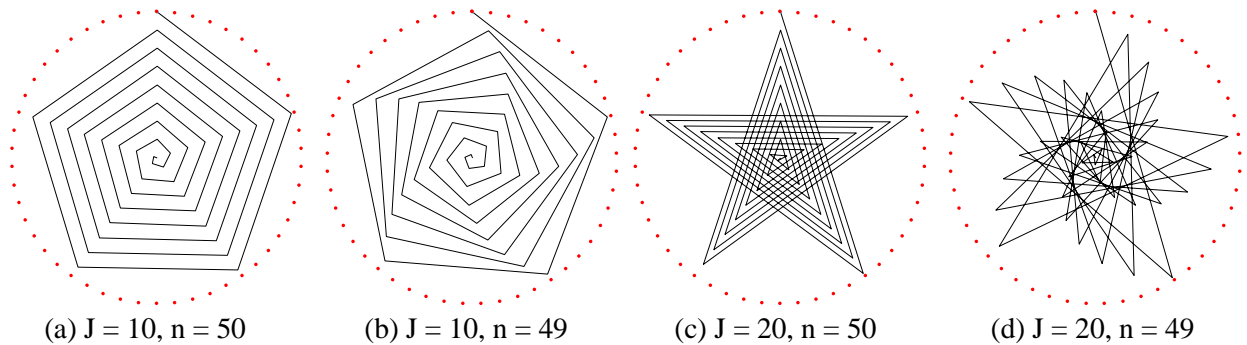
When  $r > 0$  we create spirals. Formally, the reduction in radius is  $1/r$  per jump. This means that the center is reached in  $r$  jumps, or that the image is created using  $r$  distinct connected line segments. Figure 1(a) is a classic example of an Archimedean spiral and one could readily conceptualize creating this spiral using polar graph paper and reducing the radius by a constant amount for a given angular change [2]. Common examples include watch springs or a single layer of coiled rope (sometimes called a Flemish coil). One should note that, although 1(a) looks smooth, it is actually made from 200 piecewise linear segments. Therefore, it is not formally an Archimedean spiral because the image is not a smooth curve. All images created in this file have this feature because all were created by connecting  $r$  distinct line segments rather than having a curve that passes through the vertices defining those segments.

Figure 3 provides annotated images showing per jump reductions of  $r = 2, 3, 4,$  and  $5$  for  $n = 4$  and  $J = 1$ . Since  $n = 4$ , the underlying or parent polygon is a square meaning that if  $r = 0$ , a square results. The vertices of the parent square are noted in red in each panel. The  $r$  fractional reductions in radius are labelled in each panel. The denominator in each instance is  $r$ , the number of jumps required to attain the center and the numerator counts the jump number from 1 to  $r$ . Note that in each instance, the  $r^{\text{th}}$  jump ends in the center of the parent polygon. The images in Figure 3 are also provided on a separate sheet in the *Spirals* file that can be used for classroom purposes either in overhead projection or as a handout.



**Figure 3:** Annotated images showing radius reduction per jump for four values of  $r$ .

The four images in Figure 4 have in common  $r = 40$ . Each spiral starts at the top and ends in the center of the circle which contains the vertices of the parent polygon and each counts vertex jumps clockwise starting at the top. Figures 4(a) and 4(b) have  $J = 10$  while 4(c) and 4(d) have  $J = 20$ . The “apparent” pentagon in 4(a) and pentagram in 4(c) occur because  $n = 50$  (apparent because pentagons and pentagrams require a closed circuit not an open one). The vertices of the full-sized regular polygon are shown as red dots for reference. The spiral vertices in 4(a) and 4(c) are lined up because  $50/10 = 5$ . Each of the eight apexes is on the vertical midline  $5/40 = 1/8$  closer to the center than the previous apex. (The first is only a half-apex since it is the starting point for all spirals.) Click on the *Show vertical radius* box to highlight these points.



**Figure 4:** Images created with 40 connected line segments,  $r = 40$ .

Imagine that both Figure 4(a) and 4(c) were made from spring steel bent at the appropriate places and fixed at the top. Now grab the center and tighten each coil a bit by turning clockwise. The result would be something like Figures 4(b) and 4(d) which were created with  $n = 49$ . The swirl in 4(b) occurs because the first full apex occurs  $1/8$  closer to the center AND one vertex to the right of the top vertex; five jumps have moved us one vertex beyond the top vertex because there are 49 vertices in the polygon. By contrast, the first full apex of the spiraled pentagram in 4(d) is achieved after counting  $100 = 20 \cdot 5$  vertices around. This is twice around the 49 vertex polygon with two more vertices ( $100 = 2 \cdot 49 + 2$ ) before having the fifth vertex  $1/8$  closer to the center than the half-apex at the top. With these rules in mind, go back to Figure 1 and see if you can create reasonably close to the same images as 1(b), 1(c), and 1(d).

### Clockwise Spirals versus Counter-clockwise Spirals

All images start at the top and count clockwise around the circle. This does not mean that the image must look like it was drawn this way. Figure 5 shows reflective images of Figure 4. The first segment in each panel of Figure 4 slants to the right as would be the case with a *clockwise spiral*,  $\curvearrowright$ -spiral. The first segment in each panel of Figure 5 slants to the left as would be the case with a *counter-clockwise spiral*,  $\curvearrowleft$ -spiral. Each image in Figure 5 appears to be a  $\curvearrowleft$ -spiral but it is, in fact, created using a clockwise counting process. A useful exercise is to figure out how this is accomplished. Specifically, what are the values of  $J$ ,  $r$ , and  $n$  that produce each of the images in Figure 5. Next ask: Can you describe a general rule which will produce such reflections? Does this same rule apply when  $r = 0$  and we are producing polygons or stars? If considering this last question in a classroom setting, you may want to focus the children’s attention on the number of times that you travel around the circle to create the image as noted in the Excel file. Another question to ask is: How would you have to change the description (from the start of the previous paragraph) of what you would have to do to the spring steel images in 5(a) or 5(c) to turn them into 5(b) or 5(d)?

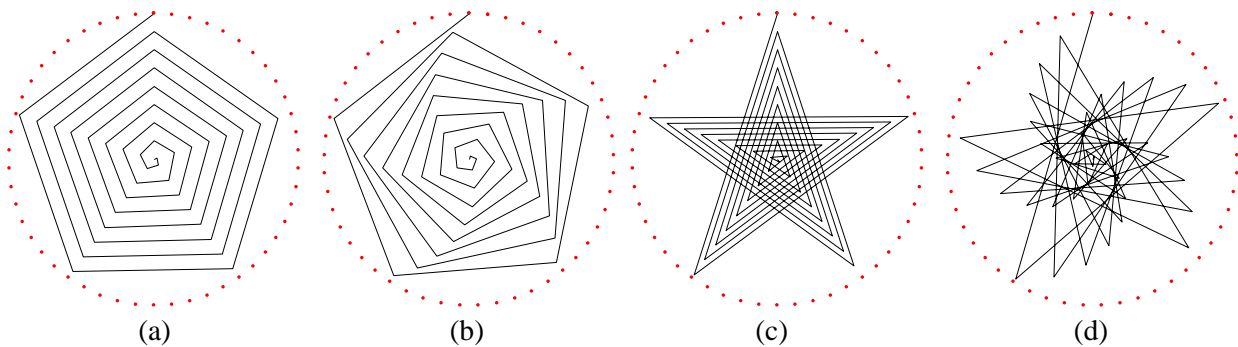


Figure 5: Reflective Images to Figure 4.

### Swirls and Flowers

By construction, all images with  $r > 0$  are piecewise linear versions of Archimedean spirals although such spirals are generally more visible when  $n$  is large and  $J$  or  $n - J$  is small. These are  $\curvearrowright$ -spirals for  $J < n/2$  and are  $\curvearrowleft$ -spirals if  $n/2 < J < n$ . Any  $(J_0, r_0, n_0)$  image has a reflective counterpart of  $(J, r, n) = (n_0 - J_0, r_0, n_0)$ .

A different question is how the image *swirls*. Figures 1(c) and 1(d) provide an initial point of reference here. Both are visually seen as  $\curvearrowright$ -spirals, but they have different swirls. Swirl in this instance refers to the rotational aspect of the image as one moves toward the center of the image. One can focus on the checkered portion or the portion that appears like you are looking downward at multiple spiral staircases. Figure 1(c) has a *clockwise swirl*,  $\curvearrowright$ -swirl, but 1(d) has a *counter-clockwise swirl*,  $\curvearrowleft$ -swirl. Figure 6 shows the four possibilities using twisted triangles. Similar images could, of course, be created for other twisted polygons. An interesting challenge question is to find general rules which produce each of these four conditions.

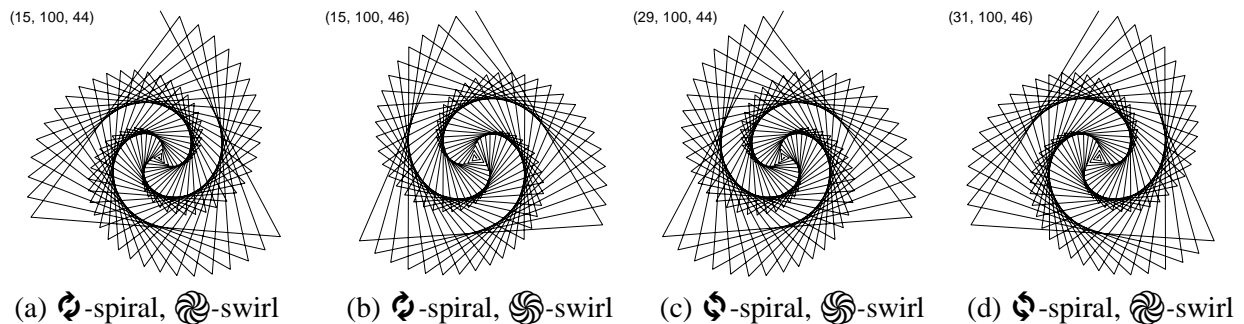
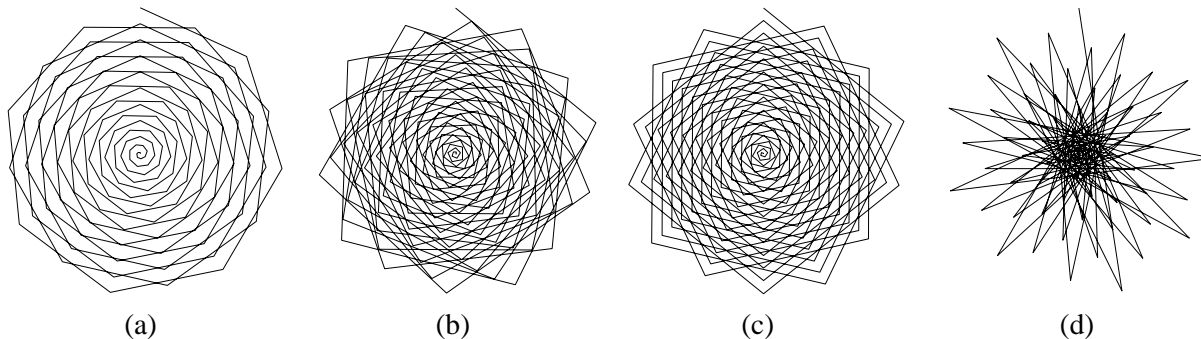





Figure 6: Comparing Clockwise and Counterclockwise Spirals and Swirls,  $(J, r, n)$  values noted.

As you click through various combinations of  $J$  and  $n$  you will inevitably encounter some that are reminiscent of flowers. Some are based on simple swirls such as 4(b) and 5(b) which are somewhat rose-like. Others, such as those shown in Figure 7, are more evocative of the numerous small petals in chrysanthemums or perhaps sunflowers with dense centers and long thin petals. Sometime these images will include a swirl, as in 7(b), but that is not required.

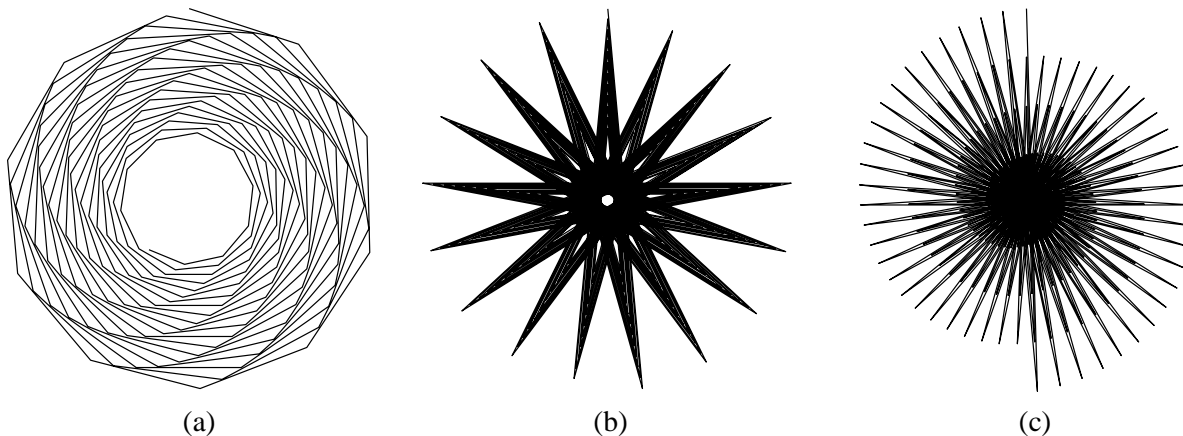


**Figure 7:** Examples of flower images.

### Additional Possibilities

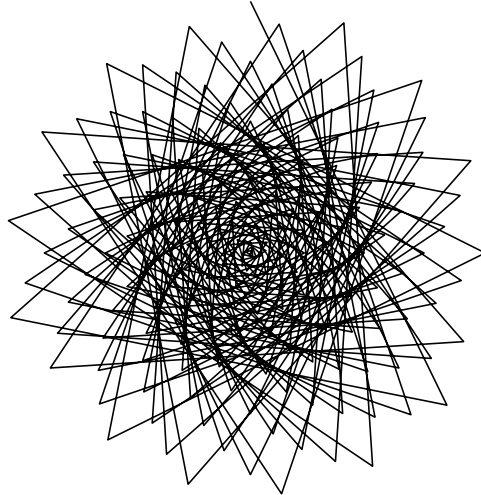
The maximum number of line segments in any spiral image is set at 200, as noted on the *Spirals* dashboard in Figure 2 above. But  $r$  is allowed to cover the range from 0 to 300 so that users can create images with open centers. The -spiral, -swirl twisted decagon in Figure 8(a) starts at the top and ends at about the “7-o’clock” location, two-thirds of the way to the center because  $r = 300$  in that image. A worthwhile exercise is to determine  $J$  and  $n$  for this image. Next ask: How can this be turned into a -swirl twisted decagon? One could also suggest that users try to create similar images based on other twisted polygons.

The eclipse starburst in Figure 8(b) is also based on  $r = 300$  but now the center is much smaller. Can you guess the values of  $J$  and  $n$  that were used to create this image? Can you create similar images with fewer or more star-points? What happens to the size of the center as the number of star-points increases? Is it possible to have an even number of star-points and still have an image similar to 8(b)? The image in 8(c) is perhaps akin to a scroll compressor which employs two interleaved Archimedean spirals.



**Figure 8:** Examples of additional image types that can be created with *Spirals Excel file*.

The final image provides an open invitation to explain a visual result. The swirls described above always appear to be either clockwise or counterclockwise but not both in the same image. The image presented in Figure 9 provides three distinct swirl patterns. The outer has 17 clockwise swirls, the middle has 14 counterclockwise swirls, and inner has 3 clockwise swirls. The obvious question is: Why?



**Figure 9:** An image with three competing swirls based on  $(J, r, n) = (17, 200, 48)$ .

### Summary and Conclusions

This workshop allows individuals the opportunity to explore spirals and swirls using the *Spirals* Excel file. This file can be used to springboard classroom discussions of fractions, symmetry, and pattern recognition as well as for independent exploration. The file is appropriate for a variety of skill levels since it is controlled by three click-driven parameters. Many patterns can be explored, even by those who have not been taught about multiplication, division, and remainders. This file offers a user experience which can be tailored to the interests and skill-level of the user. In contrast with the traditional method of creating an Archimedean spiral espoused by Hanson [2] in which one creates a single object using polar graph paper and studies it intensively, participants in this workshop spend the majority of their time exploring different geometric patterns and their numeric counterparts using a computer with Excel.

### Acknowledgements

This exposition has benefitted from discussions with Bruce Katuna and Laura Weber.

### References

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