Devising a 'Purist Knitting Aesthetic' Six-Colored Möbius Band

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Abstract

It is well-known (among mathematicians) that a Möbius band can be six-colored, unlike the plane, where four colors suffice. It is also well-known (to knitters) that a Möbius band can be knitted with intrinsic twist. This paper describes how the technical challenges to knitting a six-colored Möbius band with intrinsic twist can be overcome. Such items may be useful as manipulatives in the classroom, or as a fashion statement.

Making and Six-Coloring the Möbius Band

Take a rectangular strip of paper, give one end a 180° twist, and then sticky-tape it to the other end. The Möbius band you have made has many intriguing properties, and some of them are widely known, even to children. It has a single surface and a single edge; it's non-orientable; intriguing things happen when it is cut. (See, for example, the experiments in [10]). That its chromatic number is six is perhaps not its best-known property, but it is the one that motivated the knitting projects described in this paper.

The chromatic number of a surface is the smallest number of colors needed to color *any* map drawn on it such that no regions adjacent to each other have the same color. A plane has chromatic number four, by the famous four-color theorem. In 1910, Tietze demonstrated a map drawn on the Möbius band which required six colors (the theoretical upper bound) [11]. Placing a graph vertex in each map region, and connecting vertices if the corresponding regions are adjacent, is one way to show that the complete graph on six vertices K_6 can be embedded on the Möbius band.

Knitters are quite familiar with the term Möbius/Moebius band, often shortening it and dropping the capitalization, that is, using moebius as a common noun. They make lovely looped scarves which sit nicely and don't slide off one's shoulders. Moreover, knitting as a construction technique affords several methods [1, 2] by which the twist in the band is *intrinsic*, that is, built in as the knitting progresses. If you pause during knitting a Möbius band with an intrinsic-twist method, the object on your needles *is* a Möbius band. In the paper-and-sticky-tape method the twist is *extrinsic*. If you pause when using an extrinsic-twist method, you just have a rectangle! sarah-marie belcastro has knitted intrinsic-twist Möbius bands showing embedded K_6 [2], and instructions for six-colored Möbius band patchwork quilts are given in [10], but if anyone had knitted six-colored Möbius bands prior to 2017, they had kept that to themselves! This short paper describes the experiments and thinking that went into creating the very detailed pattern [9] which should be used if the reader wishes to make such an object for themselves.

Technical Challenges in Knitting a Six-Colored Möbius Band

Sara Jensen, in an unconventional (her own word) college course *The Mathematics of Knitting*, assigned the activity of knitting a six-colored Möbius band [6]. (Although she calls this an infinity scarf, that term can also apply to an untwisted loop.) Jensen's construction is simple: a rectangle is knitted in six colors using the *intarsia* technique of twisting yarns around each other when color changes occur within a row, *rib* fabric masking the color changes to some extent [5]. Because the rectangle is worked back-and-forth in rows, each color of yarn is to be found exactly where it is needed on the return journey along the next row. The relative dimensions of the color blocks employed in Jensen's design are exactly those of the patchwork Möbius band quilt designed by Amy Szczepański (Figure 13 of [10]), which has the benefit for

beginner knitters that the cast-on row is of a single color. Once the knitting is complete and bound off, a half-twist is introduced, and the two side edges of the rectangle are sewn together.

In my own first design for a six-colored Möbius band I wanted no sewn seam, wishing to follow a 'purist knitting aesthetic' [1]. I used a *provisional cast-on*, so that the stitches of the final row could be joined to the live stitches of the first row, by *grafting*. Grafting is done using a sewing needle, but in such a way as to join the stitches exactly as if they had been knitted. I used *garter stitch* fabric, as its two sides look identical, unlike, say, *stocking stitch* (stockinette). This gave an appropriate texture to the single-surface of the Möbius band; the color changes could have been neater, but the grafting harder, if I had chosen *seed stitch*. I did not develop a pattern, but I did document progress (Figures 1 and 2).

Figure 1: *The six-colored rectangle prior to grafting, with provisional cast-on edge at the right. Once twisted, the colors of the live stitches on the left will match that of those to which they are grafted.*

I made the design choice to include non-convex color blocks, inspired by an image from the Netherlands Wikipedia article on the four-color theorem [12]. Topologically, this is the same as Tietze's map and the one in [10]; they comprise three internal pentagons and three hexagons. Each color block shares an edge with the other five. The edge of the Möbius band is made from the left-over (sixth) edges of the three hexagons; in my design, this has been worked as a *slipped-stitch edging*, for an elegant finish.

Figure 2: *First finished six-colored Möbius band (September 2017). It is made from 8 ply (DK) pure Australian wool and is large enough to wear as a snug cowl. (18 cm by 60 cm.)*

Both Jensen's design and mine utilize the fact that knitted fabric is the same color all the way through, unlike the patchwork Möbius band quilts of Szczepański [10], which must comprise two layers of fabric with their wrong sides together. However, the twist is extrinsic, and the glorious possibility to knit it intrinsically has not been realized.

I next learned the popular *Möbius cast-on* of Cat Bordhi [3] using one color, confirming what I had suspected: it would be difficult to use this cast-on to create a six-colored Möbius band, because of the need to use several colors of yarn around the central circle or 'equator' of the band. One cannot pause when using this cast-on, to change colors; constant tension and motion is the key to it. However, the technique devised by Miles Reid [2, 8] was adaptable to my needs. In this method, the Möbius band develops above and below a provisional scrap-yarn cast-on (visible in Figure 3 (a)), ultimately removed.

However, two more challenges needed to be overcome. First, when knitting in the round, blocks of color are not trivial to knit. As a [right-handed] knitter leaves a color block, that color of yarn remains at the *left-hand* side of the block. When the block is reached again, however, the yarn is required at the *right-hand* side. Knitting a Möbius band with intrinsic twist is still knitting in the round, in fact a sort of double round, above and below the equator. Therein lies the second challenge: the three internal blocks of color are built partially above and partially below the equator. Were a single piece of yarn of each color being used, upon coming to a color block the required yarn would be diagonally opposite to where it was wanted. In the piece *Dusk or Dawn*, which uses her pattern *K_6-on-a-Möbius-band scarf*, sarah-marie belcastro reports using separate pieces of yarn for the colors in each round, resulting in 68 ends to weave in [2]! I was energized to sort out these remaining challenges when Moira Chas [4] showed a crocheted six-colored Möbius band. Non-working yarn can be hidden inside crochet stitches, in a technique called *tapestry crochet*, so that colors can be carried along to where they are next needed. While I quite like crochet, I wanted to *knit* a six-colored Möbius band, using the 'pur[e]st knitting aesthetic' [1] possible!

Investigation led me to *intarsia-in-the-round*. This technique involves working rounds somewhat like rows, turning after anchoring the yarn around a stitch, and working back in the opposite direction (*short rows* with *wrap and turn*). Colors are then in place where needed, and the integrity of the 'round' structure of the object is maintained. I practised by making an untwisted item (a hat). The position at which the turn occurs was not invisible, but by doing it at the point of one of the color changes in each round I expected it to be less noticeable. By thought experiments and diagrams, I realized that if the three internal color blocks were worked with separate skeins above and below the equator (see Figure 3(a)), the final challenge could be overcome.

(a) (b)

Figure 3: *(a) The white stitches are the provisional scrap-yarn cast on (equator) and the two skeins of dark green yarn are visible at lower right;*

(b) Möbius cowl with Tietze's six-coloring in shades of green. (14 cm by 86 cm.)

With six colors, the number of yarn ends to be woven in at completion could theoretically be as low as 12. My first pattern, shown in Figure 3(b), had 20, which is a lot less than 68, but since so many design challenges had been overcome, I wanted to solve this one as well. After all, Sabetta Matsumoto solved an even more difficult one, in grafting double-knit cellular automata Möbius scarves [7]. The final pattern [9], used to create the capelet shown in Figure 4, satisfyingly has only 12. This was achieved by working from the center of the skein of yarn for the three internal color blocks.

Summary and Conclusions

Whether one wants a sturdy class-room manipulative or an unusual garment, I offer my solution to the challenges of knitting, with intrinsic twist, a six-colored Möbius band. In the objects shown in Figures 3

and 4, the same mass (or length) of yarn has been used to create each of the color blocks. Perhaps you would like to make one in which the color blocks have the same width, rather than the same area.

Figure 4: *Capelet made in 50% acrylic, 50% tensel yarn. Note the small number of yarn ends visible in this final design, awaiting their weaving-in. A narrow i-cord bind-off has been used. (34cm by 135 cm.)*

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