Quadruple Tetrahedron Surface Tilings

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Abstract

This paper introduces a way to create monohedral tilings of the surface of a regular tetrahedron. We design a special plane diagram (or pattern) that can be printed on a single sheet of paper and folded up to be a regular tetrahedron. In fact, if a diagram is a net of a regular tetrahedron, then we can use 4 copies of it to construct another new net of a regular tetrahedron; that is, a tiling of the surface of a regular tetrahedron. Finally, by similar construction modes, we can obtain different surface tilings of a regular tetrahedron with 16 copies of it, 64 copies of it, and so on. Thus we have new procedures for creating artistic surface tilings of a regular tetrahedron.

Definitions of net

Traditionally, a *net* of a polyhedron refers to a (plane) diagram in which the edges of a polyhedron are shown, and we can fold up the net along the edges of such a polygon. However, our definition of net in this paper isn't limited by that we have to fold up along all the edges. Moreover, the net itself does not need to be a polygon.

Our example net here is shaped like the Greek letter π, in celebration of Pi (π) day (Figure 1).

(a)   (b)   (c)
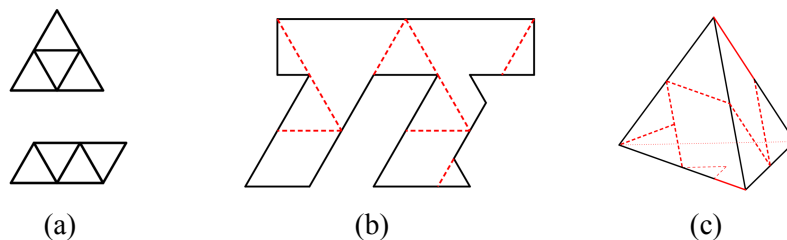
Figure 1: *Nets of a regular tetrahedron*：
*(a) traditional nets, (b) the π-net, (c) folding up π*

Creating a Net

The first time we learned about nets of tetrahedrons was in page 41 of Akiyama's book [1]. Akiyama provided several kinds of nets of tetrahedrons, in which some are irregular, and some are artistic diagrams. We were very interested in how to make a special net like that. After lots of studying, we have found a method by which everyone can easily design a net by him/herself.

Before we start creating such a net, we designd a special four-colored tiling in own term used here, called *hexagram floor*. A hexagram floor is a coloring method of equilateral triangle tessellation, which satisfies the following two conditions: first, the area of 4 unit triangles should be the same as the net you want to create later; second, every six unit triangles around a hexagon which contained 6 unit triangles have to be the same color (for easy to read in black and white printing, we adjust the intensity of the colors), as shown in Figure 2.
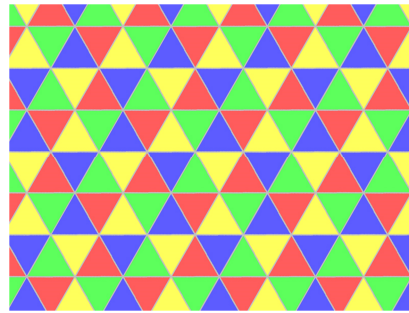
Figure 2: *Hexagram floor*

We are now ready to design the net. First, let's put a traditional net of a regular tetrahedron on the hexagram floor by using the triangle-type or the parallelogram-type (the region bounded by thick line segments). It covers four unit triangles of different colors exactly. Then, try to remove a piece inside the net from one triangle to another triangle of the same color in the relative position. If the direction of the goal triangle you wish to remove to is different, just rotate your piece by 180 degrees. Figure 3 shows that one translation and three rotation moves are needed to create the π-net.
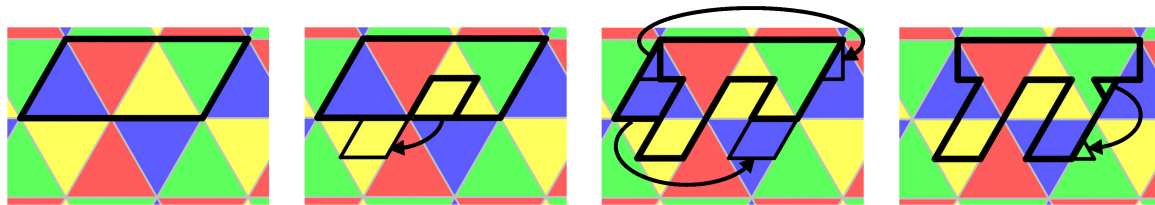
Figure 3: *Creating the net step-by-step*

The reason is that the four colored triangles will be the four faces of the regular tetrahedron you want to fold up. If you translate a piece from one triangle to another, the piece will still be at the same position of the tetrahedron. It means that if a diagram is laid on the floor, that each part in the triangle of the same color can exactly be combined to be a unit triangle completely. Then it is a net of a regular tetrahedron. We cannot ensure whether all diagrams produced by this way can be folded up to a tetrahedron. It needs to be proved.

The following are some nets we made (Figure 4).

(a)   (b)   (c)
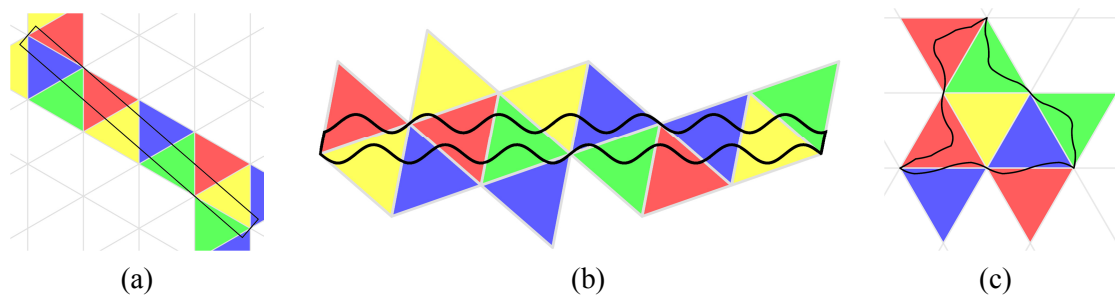
Figure 4: *Some examples of nets*：

(a) a rectangle, (b) a sine wave ring, (c) a "cat" designed by using Escher's method

Here are some suggestions for easier designing. If you want to make a net by yourself, we suggest that you can print a triangle grid, and then use 4, 9, or any square number of triangles as the unit triangle of the hexagram floor. For example, our π-net is based on a 4-triangle-unit floor.

Tiling and Quadrupling the Net

A theorem proved in page 35 of [1] by Akiyama asserts that any net of an isohedral tetrahedron can tile the plane. In Figure 5, the π-nets form a tessellation of the plane. This is where our idea of the hexagram floor came from, and we will use this idea to tile the surface of a regular tetrahedron next.
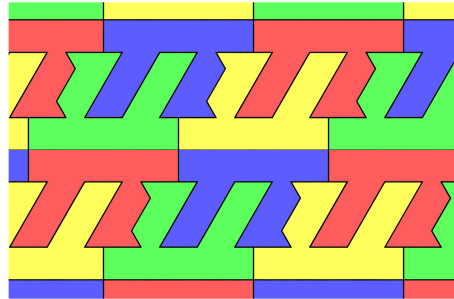

Figure 5: *A tiling by the π-net*

The π-tiling in Figure 5 is a p2 tessellation of the plane, which means that there is a 180-degree rotation inside the pattern, without axial symmetry. We found that four adjacent copies of π can form a parallelogram-like shape, and a parallelogram is one of the traditional nets of a regular tetrahedron. So we thought, if we double-sized the hexagram floor, will four copies of π still be a net of a double-sized tetrahedron?

Let's take a look at Figure 6. We put four copies of π on the hexagram floor first. And since every four adjacent unit triangles can be combined to a double-sized triangle, we can recolor the floor, and then a new double-sized hexagram floor has produced. Hence, four copies of π just satisfies the condition to be a net.
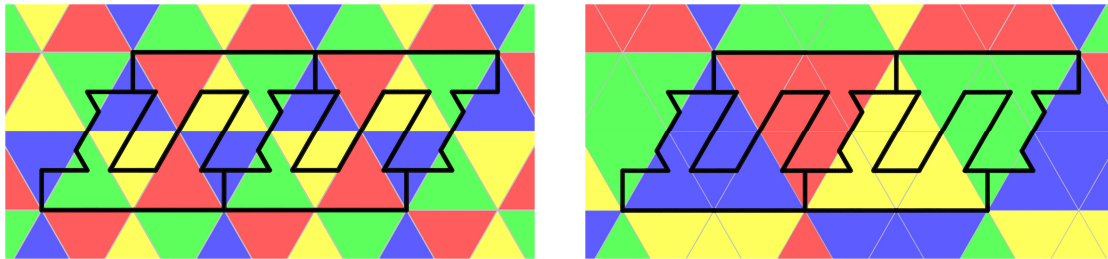

Figure 6: *Double-sized net on the double-sized hexagram floor*

Infinity Patterns

By using the quadrupling method again and again, if we design a π-net, then four copies of π will also form a net. The same is true for 16 copies of π, 64 copies of π, and so on. It means that if a diagram is a net of a regular tetrahedron, we can use 4^n copies of it, where n is a positive integer, to make a surface tiling of a regular tetrahedron based on the concept of mathematical induction. We illustrate tilings with π, 4 copies of π, 16 copies of π, and 64 copies of π of a regular tetrahedron, as shown in Figure 7.
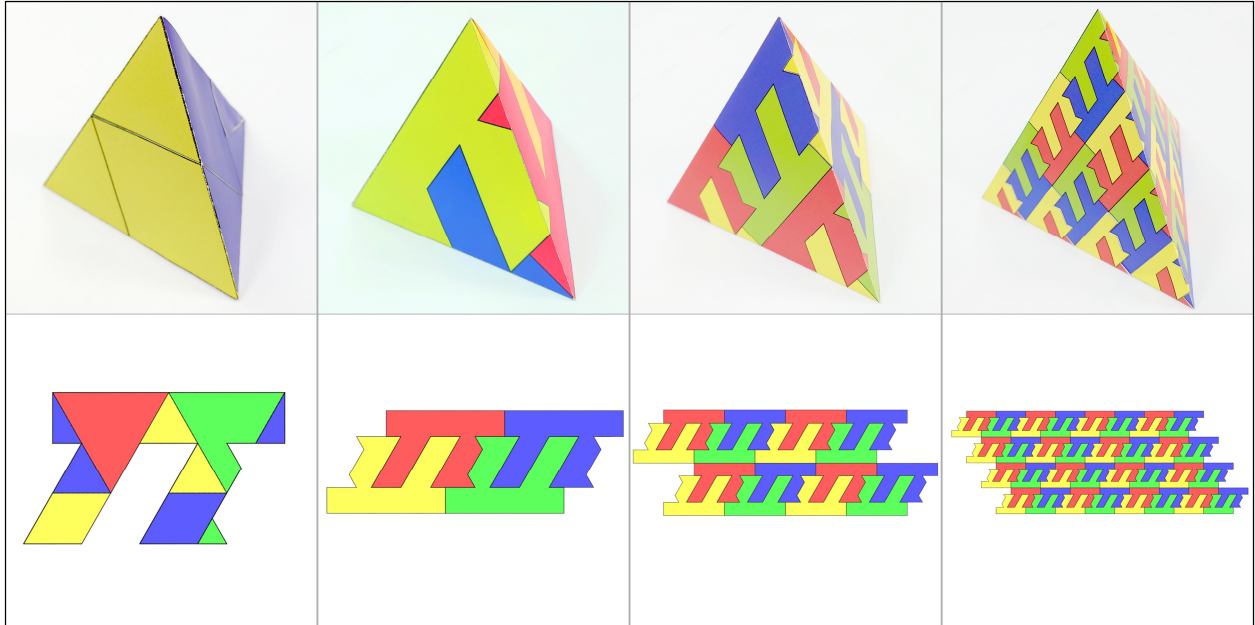
Figure 7: *Surface of regular tetrahedrons tiled by 4^n copies of π*

Summary and Conclusions

We found an easy way to design an artistic diagram to tile the surface of a regular tetrahedron. By the hexagram floor to change a traditional net to an artistic net, we can combine four copies of an artistic net to a new one, and so on. Besides, we also wonder whether we can use similar methods to handle other Platonic solids, for making an artistic net or extending a net to a tiling. Maybe there exists an elegant method to do it.

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References

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